Stop-loss premiums for $X$ at retention $d$ satisfy $E[X - d | d] = \int_d^\infty [1 - F_X(z)] \, dz$.
Stop-loss transform: $\pi_X(d) := E[(X - d)_+]$, $-\infty < d < \infty$.
So $\pi'_X(d) = F_X(d) - 1$.
Optimality of stop-loss: Let $I(X)$ be the insurance payment for loss $X \geq 0$, and $0 \leq I(z) \leq z$ ∀$z$.
If $E[I(X)] = E[(X - d)_+]$, then $\text{Var}[X - I(X)] \geq \text{Var}[X - (X - d)_+]$.

## Individual model

### Mixed random variable
If $Z = IX + (1 - I)Y$ with $I \sim \text{Bernoulli}(p)$ independent of $Y$ and $Z$, then $F_Z(z) = p F_X(z) + (1 - p) F_Y(z)$.

Riemann-Stieltjes integral $E[g(Z)] = \int g(z) \, dF_Z(z) = \int g(z) [F_Z(z) - F_Z(z - 0)] + \int g(z) F_Z'(z) \, dz$.
Useful for mixed discrete/continuous r.v.'s $Z$.

### Convolution
$F_X \ast F_Y(z) \overset{def}{=} \int F_Y(s - x) \, dF_X(x)$.
So if $X$ and $Y$ are independent, then $X + Y \sim F_X \ast F_Y$.
For continuous densities $f_X \ast f_Y(s) = \int f_X(x) f_Y(s - x) \, dx$; for discrete densities $F_X \ast F_Y(s) = \sum f_X(x) f_Y(s - x)$.
Convolution powers $F^{*n}(s) \overset{def}{=} F \ast F \ast \cdots \ast F$ ($n \geq 1$ times), and $F^{*0}(s) \overset{def}{=} 1$ if $s \geq 0$, zero otherwise.

### Transforms
Moment generating function (mgf): $m_X(t) = E[e^{tX}]$, $-\infty < t < h$, for some $h > 0$.
Cumulant generating function (cgf): $\kappa_X(t) = \log m_X(t)$.
Probability generating function (pgf): for natural-valued $X$, $g_X(t) = E[t^X] = \sum t^k \Pr[X = k] = m_X \log(t)$.
Characteristic function: $\phi_X(t) = E[e^{itX}] = m_X(it)$.

‘Recognize the mgf!’ If $m_X = m_Y$, then $F_X = F_Y$.
The other transforms also have this 1–1 relation.

### Generating moments
As $m_X(t) = \sum E[(tX)^k] / k!$, the coefficient $m_X^{(k)}(0) = \frac{d^k}{dt^k} m_X(t) |_{t=0}$ is the $k^{th}$ moment of $E[X^k]$, $k = 0, 1, \ldots$

### Cumulants
$\kappa_X^{(k)}(0)$ generates the first 3 cumulants
$\kappa_1 = E[X] = \mu$, $\kappa_2 = E[(X - \mu)^2] = \sigma^2$, $\kappa_3 = E[(X - \mu)^3]$.
Skewness $\gamma_X = \kappa_3 / \sigma^3$.
Coefficient of excess $\varepsilon = \frac{(X^2 - \mu^2)}{\sigma^2} - 3$ (‘kurtosis’).

## Compound r.v.
$S = X_1 + \cdots + X_n$ with independent $X_i$;
write $E[S] = \mu$, $\text{Var}[S] = \sigma^2$, $\gamma = \gamma > 0$ and $Z = \frac{S - \mu}{\sigma}$.
Central Limit Theorem: $Z \overset{\text{indep.}}{\sim} \text{Normal}(0, \sigma^2)$, so $F_Z(s) \approx \Phi(s; \mu, \sigma)$.

### Transformed Gamma
Translated Gamma: $Z \overset{\text{indep.}}{\sim} \text{gamma}(a, \gamma)$ if $X \sim \text{gamma}(a, \gamma)$. So $F_S(s) \approx G(s - x, \alpha, \beta)$ with $G$ the gamma cdf and 3 fitted moments, i.e. $\alpha = \frac{1}{\beta^2}$, $\beta = \frac{\gamma}{a}$, $x = \mu = \frac{\gamma}{a^2}$.
Normal Power: $Z \overset{\text{indep.}}{\sim} U \sim \text{gamma}(a, \gamma) = \gamma T^2 - 1$ with $U \sim \text{normal}(0, 1)$, so

\[
\text{for quantiles: } \Pr[Z \leq s] = \Phi(s; \mu, \sigma^2) \approx p = \Phi(s^2 - 1); \\
\text{for cdf-values: } \Pr[Z \leq s] \approx \Phi\left(\frac{s}{\sqrt{\sigma^2 + \frac{\mu^2}{2} + 1 - \frac{2}{3}}}\right).
\]

## Collective model

### Compound r.v.
$S = X_1 + \cdots + X_N$ with $X_1, X_2, \ldots$ iid ~ independent of $N$.
Notation: $\mu_X = E[X^2]$, $P(s) = \Pr[X \leq s]$, $F(s) = \Pr[S \leq s]$.
Then $E[S] = E[N \mu_X]$, $\text{Var}[S] = E[N] \text{Var}[X] + \text{Var}[N] \mu_X^2$, and $\text{mg}(t) = m_X \log(m_X(t))$.
$N \sim \text{Poisson}(\lambda)$: $E[S] = \lambda \mu_X$, $\text{Var}[S] = \lambda \mu_X^2$, $m_X(t) = e^{t \mu_X - t \lambda}$.
Compound r.v. formula: $f_X(s) = \sum_{n=0}^{\infty} m_X^n(t) \Pr[N = n]$.

### Number of claims
$N \sim \text{Poisson}$. $S$ has a Poisson process (see below).
Parameter uncertainty: If $N \mid \lambda = \lambda \sim \text{Poisson}(\lambda)$ and $\lambda \overset{\text{indep.}}{\sim} \text{gamma}(a, \gamma)$, then $N \sim \text{NegBin}$.

Cumulative: If $M = L_1 + \cdots + L_N$ with $N \sim \text{Poisson}$ and $L_i$ iid ~ lognormal(c) (so $\Pr[L = k] \approx \frac{1}{\sqrt{2\pi c^2}} \frac{1}{k!} e^{-c^2 / 2}$, $k = 1, 2, \ldots$), then $M \sim \text{NegBin}$.
Limit: For $\lambda = (1 - p) / p$ and large $r$, $\text{NegBin}(p, r) \approx \text{Poisson}(\lambda)$.

### Properties of compound Poisson (CP) r.v.'s
If $S_j \overset{\text{indep.}}{\sim} \text{CP}(\lambda_j, P_j)$ and $\lambda_j \sim \text{gamma}(\lambda_j)$, then $\sum S_j \overset{\text{indep.}}{\sim} \text{Poisson}(\lambda_1 + \lambda_2 + \cdots + \lambda_n)$.

If $N_j \sim \text{Poisson}(\lambda_j)$, then $\sum S_j N_j \overset{\text{indep.}}{\sim} \text{CP}(\lambda_1, P_1 \cdots P_n)$.

If $S = X_1 + \cdots + X_N \overset{\text{indep.}}{\sim} \text{CP}(\lambda, P)$ and the frequency of amount $x_j$ in $S$ is $N_j = \#(x_j \in S)$ (and therefore $S = \sum x_j N_j$), then $N_j \sim \text{Poisson}(\lambda x_j)$ independent, $j = 1, 2, \ldots$.

If $X$ is integer, with $x_j = j \ \forall j$ (so $S = \sum x_j N_j$), the db of $S$ can be computed by successively convolving the density of $1N_1 + 2N_2 + \cdots + (k-1)N_{k-1}$ with the (sparse) one of $kN_k$, $k = 2, 3, \ldots$ (Sparse vector algorithms)
Panjer’s recursion If $N \sim \text{Poisson}(\lambda)$ and $X \in \{0, 1, \ldots\} \sim p$, then $f(s) = \Pr(S = s)$ can be computed recursively:

- starting value $f(0) = e^{-\lambda(1-\rho(0))}$;
- recursion formula $f(s) = \frac{\lambda}{s!} \sum_{h=0}^{s} \rho(h) f(s-h)$, $s = 1, 2, \ldots$

If $N \sim \text{Bin}(r, p)$ take $a = p/(1-p)$, $b = -a(r+1)$; if $N \sim \text{NegBin}(p, r)$ take $a = 1 - p$, $b = a(r-1)$.

Next: $f(0) = \frac{\Pr[N = 0]}{\bar{m}(0)}$ if $p(0) = 0$; $f(s) = \frac{\rho(s)}{\bar{m}(0)}$ if $p(s) > 0$; $f(s) = \frac{1}{\lambda - \rho(s) - s} \sum_{h=1}^{s} \left( \frac{a + b + 1}{\rho(h)} \right) \rho(h) f(s-h)$, $s = 1, 2, \ldots$

Stop-loss premiums: $\bar{m}(k) = \bar{m}(k-1) + \left[ -1 - F(k-1) \right]$ with $p(0) = [E[N]E[X], \text{for integer } k, x, k.$

Approximations Use CLT, Translated Gamma and NP for compound r.v.s.’ with large $E[X]$ as well as.

For $c \rho(\lambda, X)$, use $\mu = E[X], \sigma = E[X]^2$, and $\gamma = \text{Var}[X]/\sigma^3$.

If cumulants $x$ of $X$ are known, use x as a real.

$E[X] = x_1, E[X^2] = x_2 + x_1^2$ and $E[X^3] = x_3 + 3x_2x_1 + x_1^3$.

Fast Fourier Transform method to approximate compounding $(f_0, f_1, \ldots, f_n) \rightarrow \frac{1}{n} \sum_{k=0}^{n-1} \rho(T^k)\hat{f}(k)$.

The matrices $T = \text{discrete Fourier transforms, } n$ is so large that $\Pr|S| \approx 0$, and $g_N(x)$ is the pdf of $N$.

Collective model approach If the payment $b_i$ of a claim is a constant $i/p x_i$ is fixed, the distribution of $S$ is $\sum_i b_i$, with $\Pr[S_i = 1] = x_i = 1 - \Pr[S_i = 0]$.

Approximate $F_S$ by $F_B$ with $S = \sum N_i b_i$ and $N_i \sim \text{Poisson}(\lambda_i)$.

Taking $1 - e^{-\lambda x} = x$, $\forall x$ gives an open collective model.

The collective model has $\lambda = \sum_{i} \lambda_i$.

Then $S \sim cP(\gamma, p(b_i) = q_i/q_x)$.

$S \sim \text{Po}(Z_1, \text{with } Z_1 \sim cP(1, k_i b_i))$.

$S \sim cP(n_1, \frac{\rho(s)}{\bar{m}(0)} \Pr(S = s), s \leq b_i)$, and $\bar{E}[S] = E[\bar{S}]$ but $\text{Var}[S] > \bar{V}[S]$.

Random claim amount: replace $\bar{S} = \sum S_i$, $S = \sum S_i$, $\sum_{i=1}^{N} X_i(j)$ with $N_i \sim \text{Poisson}(a)$, $X_i(1) \sim X_i$, all independent.

Loss distributions For claim totals, consider $S = X_1 + \cdots + X_N$ with $N \sim \text{Poi}(\lambda)$ or (Neg)Bin, or some other counting db.

Generating samples from a distribution
1) Inversion method: if $U_i \sim \text{Uniform}(0, 1)$, iid, then $F^{-1}(U_i)$, $i = 1, \ldots, n$, is a sample from $F$.
2) Transform from another distribution, e.g. $-\log U \sim \text{exp}(1)$.
3) Rejection method: draw a ‘simpler’ r.v. and accept it with the right probability.

Maximum likelihood estimation Sample loglikelihood is $\ell(\alpha, \beta, \ldots, \gamma) = \prod \ell(Y_i; \alpha, \beta, \ldots)$.

To maximize $\ell$, solve the ML-equations $\partial \ell/\partial \theta = 0, \ldots$ to get

1) analytical solutions (Poisson, Bin, (log-normal), IG, Pareto);
2) partial solutions using which only a one-dimensional optimization is needed (NegBin, gamma);
3) no helpful solution: use a multidimensional optimization routine.

Mixture/combination of exponentials If $X, Y \sim \text{Exp}(\lambda)$, then $X + Y \sim \text{Exp}(\lambda + \lambda)$.

Risk measures Value-at-Risk $\text{VaR}(S; p) = \frac{1}{p} \text{VaR}_{-1}(p)$.

Holding capital $d = \text{VaR}(S; 1 - d)$ minimizes the value of cost plus expected cost of overshoot $i \cdot d + \Pr(S < d)$.

VaR is not subadditive, since $\text{VaR}(S + T; p) > \text{VaR}(S; p) + \text{VaR}(T; p)$ may hold.

Expected shortfall $\text{ES}(S; p) = \frac{1}{1 - p} \text{ES}(S; 1 - p)$.

Tail-Value-at-Risk $\text{TVaR}(S; p) = \frac{1}{1 - p} \text{TVaR}(S; 1 - p)$

Aka Conditional Value-at-Risk (CVaR).

Conditional Tail Expectation $\text{CTE}(S; p) = \mathbb{E}[S | S > \text{VaR}(S; p)]$.

Aka Conditional Value-at-Risk (CVaR).

Continuous r.v.s.’ only.

Bonus-malus systems Lomirantza efficiency If $b(\lambda)$ is the steady-state premium for claim frequency, the efficiency is $\lambda \frac{\partial \lambda}{\partial b} \frac{b(\lambda)}{\lambda}$.

So $b(\lambda(1+h)) \approx b(\lambda)(1 + e(\log \lambda))$. 

Ruin theory Poisson process $N(t)$ is a Poisson process if for all $t > 0, h > 0$ and each history $N(s), s \leq t$, its increments satisfy:

$N(t + h) - N(t) \sim \text{Poisson}(\lambda h)$.

The increments are independent and stationary; the process is memoryless.

The infinitesimal increments of a Poisson process have

$\lambda \mathbb{E}[T | N(s), 0 \leq s \leq t] = \mathbb{E}[T] = \lambda dt$ if $k = 0$.

$1 - \lambda dt$ if $k = 1$ + $O(dt^2)$.

$0$ if $k = 2, 3, \ldots$.

If $T_i$ are times of an occurrence, then the waiting times $T_i = T_1, W_j = \sum_{j=1}^{i} T_j, J > 1$ are iid and exponential distribution.

Classical ruin process $U(t) = u + c - S(t), t \geq 0$, where $U(t)$ is the insurer’s random capital at time $t$.

$u = U(0)$ is the initial capital;

c = the (constant) premium income per unit of time;

$S(t) = X_1 + X_2 + \cdots + X_N(t)$, with $N(t)$ number of claims till $t$, which is a Poisson process, and $X_i$ the size of the $i^{th}$ claim, assumed non-negative.

Ruin probability is $Pr(U < t), \text{ the time of ruin } T$ in process $U(t)$ is the defective r.v.

$T = \min\{t \geq 0 \& U(t) < 0\}$

Adjustment coefficient is the unique $R > 0$ solving $1 + (1 + \theta)E[X]R = \text{MST}(R)$.

Here $0$ is the loading factor, so $c = (1 + \theta)E[X]$. If $S$ is the total claims in a unit of time, then $R$ also uniquely solves $e^{-R} = E[RS] \iff \text{MST}(R) = 1 \iff e^{-R} = \frac{1}{R} \log MS(R)$.

So if $c$ is an exponential annual premium, the risk aversion is $R$.

Lundberg inequality $\psi(u) < e^{-Ru}$. Always $\psi(0) = 1$.

For exponential($\beta$) claims, $R = \frac{1}{\beta^2}$ and $\psi(u) = e^{-Ru}$.

Discrete time ruin model Let $G_n = c - (S(n) - S(n-1))$ be the gain in year $n$, so $U(n) = u + G_1 + \cdots + G_n, n \geq 1, 2, \ldots$ Define discrete ruin time $\bar{T} = \min\{n : U(n) < 0\}$, ruin probability $\psi(u) = Pr(\bar{T} < \infty)$, adjustment coefficient $\bar{R}$ solving $\text{MST}(\bar{R}) = 1$.

In a Poisson process $\bar{R} = R$ but $\bar{\psi}(u) = \psi(u)$ $\forall u$.

If $G_n \sim N(\mu, \sigma)$, then $\bar{R} = 2\mu/\sigma^2$.

Lundberg inequality: $\psi(u) < e^{-Ru}$.