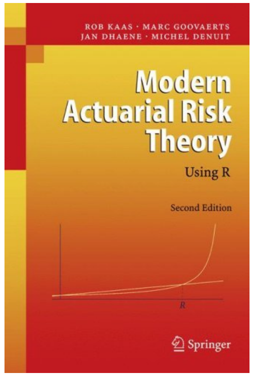
Modern Actuarial Risk Theory Cheatsheet



Created in honor of Marc Goovaerts and presented to him at the Memorable Actuarial Research Conference, June 13, 2011

Utility theory

Zero-utility premiums [aka *Equivalent utility premiums*]

An *insured* with utility $u(\cdot)$ and wealth w will pay a premium π or less to insure loss X if π solves $\mathrm{E}[u(w-X)] = u(w-\pi)$.

An *insurer* with utility $u(\cdot)$ and wealth w needs π or more to cover X if $\mathrm{E}[u(w+\pi-X)] = u(w)$.

Jensen's inequality $\mathrm{E}[v(Y)] \geq v(\mathrm{E}[Y])$ for any convex $v(\cdot)$ and Y.

If u has *decreasing marginal utility* (so u' decreases) it is called *risk averse*, since then $P \geq \mathrm{E}[X]$.

Arrow-Pratt approximation If $\mathrm{E}[X] = \mu$, $\mathrm{Var}[X] = \sigma^2$, then $P \approx \mu + \frac{1}{2} r(w-\mu)\sigma^2$.

Here $r(w) = -u''(w)/u'(w)$ is the *absolute risk aversion coefficient*.

Utility function	$u(w) =$	range, parameter
Linear	w	
Quadratic	$-(\alpha - w)^2$	$w \leq \alpha$
Logarithmic	$\log(\alpha + w)$	$w > -\alpha$
Exponential	$-\alpha e^{-\alpha w}$	$\alpha > 0$
Power	w^c	$w > 0,\ 0 < c \leq 1$

Note: $u(\cdot)$ is equivalent with $au(\cdot) + b$ if $a > 0$.

Exponential premiums If $u(\cdot)$ exponential, $P = \frac{1}{\alpha}\log\left(\mathrm{E}[e^{\alpha X}]\right)$.

$P(\alpha)$ increases from $P(0+) = \mathrm{E}[X]$ to $P(\infty) = \max[X]$.

The risk aversion $r(w) \equiv \alpha$ is constant.

Stop-loss premiums for X at retention d satisfy $\mathrm{E}[(X-d)_+] = \int_d^\infty [1 - F_X(x)]\,\mathrm{d}x$.

Stop-loss transform: $\pi_X(d) := \mathrm{E}[(X-d)_+]$, $-\infty < d < \infty$.

So $\pi'_X(d+0) = F_X(d) - 1$.

Optimality of stop-loss: Let $I(X)$ be the reinsurance payment for loss $X \geq 0$, and $0 \leq I(x) \leq x\ \forall x$.
If $\mathrm{E}[I(X)] = \mathrm{E}[(X-d)_+]$, then $\mathrm{Var}[X - I(X)] \geq \mathrm{Var}[X - (X-d)_+]$.

Individual model

Mixed random variable If $Z = IX + (1-I)Y$ with $I \sim$ Bernoulli(p) independent of Y and Z, then $F_Z(z) = pF_X(x) + (1-p)F_Y(z)$.

Though $\mathrm{E}[Z] = p\mathrm{E}[X] + (1-p)\mathrm{E}[Y]$, still $Z \nsim pX + (1-p)Y$.

Riemann-Stieltjes integral $\mathrm{E}[g(Z)] \overset{\text{not.}}{=} \int g(z)\,\mathrm{d}F_Z(z) \overset{\text{def}}{=} \sum g(z)[F_Z(z) - F_Z(z-0)] + \int g(z) F'_Z(z)\,\mathrm{d}z$.

Useful for mixed discrete/continuous r.v.'s Z.

Convolution $F_X * F_Y(s) \overset{\text{def}}{=} \int F_Y(s-x)\,\mathrm{d}F_X(x)$.

So if X and Y are independent, then $X + Y \sim F_X * F_Y$.

For continuous densities $f_X * f_Y(s) = \int f_X(x) f_Y(s-x)\,\mathrm{d}x$;

for discrete densities $f_X * f_Y(s) = \sum f_X(x) f_Y(s-x)$.

Convolution powers $F^{*n} \overset{\text{def}}{=} F * F * \cdots * F$ ($n \geq 1$ times), and $F^{*0}(s) \overset{\text{def}}{=} 1$ if $s \geq 0$, zero otherwise.

Transforms

Moment generating function (mgf): $\mathrm{m}_X(t) = \mathrm{E}\left[e^{tX}\right]$, $-\infty < t < h$, for some $h > 0$.

Cumulant generating function (cgf): $\kappa_X(t) = \log \mathrm{m}_X(t)$.

Probability generating function (pgf): for natural-valued X, $g_X(t) = \mathrm{E}[t^X] = \sum t^k \Pr[X = k] = \mathrm{m}_X(\log t)$.

Characteristic function: $\phi_X(t) = \mathrm{E}[e^{itX}] \overset{?}{=} \mathrm{m}_X(it)$.

'Recognize the mgf' If $\mathrm{m}_X = \mathrm{m}_Y$, then $F_X = F_Y$.

The other transforms also have this 1–1 relation.

Generating moments As $\mathrm{m}_X(t) = \sum \mathrm{E}[(tX)^k]/k!$, the coefficient $\mathrm{m}_X^{(k)}(0) = \frac{\mathrm{d}^k}{\mathrm{d}t^k}\mathrm{m}_X(t)\big|_{t=0}$ of $t^k/k!$ in the power series for the mgf is the k^{th} moment $\mathrm{E}[X^k]$, $k = 0, 1, \ldots$.

Cumulants $\kappa_k \overset{\text{def}}{=} \kappa_X^{(k)}(0)$ generates as first 3 cumulants $\kappa_1 = \mathrm{E}[X] = \mu$, $\kappa_2 = \mathrm{E}[(X-\mu)^2] = \sigma^2$, $\kappa_3 = \mathrm{E}[(X-\mu)^3]$.

Skewness $\gamma_X \overset{\text{def}}{=} \kappa_3/\sigma^3$.

Coefficient of excess $\overset{\text{def}}{=} \kappa_4/\sigma^4 = \mathrm{E}\left[\frac{(X-\mu)^4}{\sigma^4}\right] - 3$ *('kurtosis')*.

Transforms and convolution Suppose X, Y are *independent*.

Then $F_{X+Y} = F_X * F_Y$ and $\mathrm{m}_{X+Y} = \mathrm{m}_X \cdot \mathrm{m}_Y$.

Also $g_{X+Y} = g_X \cdot g_Y$ and $\phi_{X+Y} = \phi_X \cdot \phi_Y$.

For the cgf, $\kappa_{X+Y} = \kappa_X + \kappa_Y$.

Cumulants cumulate: $\kappa_j(X+Y) = \kappa_j(X) + \kappa_j(Y)\ \forall j$.

Approximations Let $S = X_1 + \cdots + X_n$ with independent X_i; write $\mathrm{E}[S] = \mu$, $\mathrm{Var}[S] = \sigma^2$, $\gamma_S = \gamma > 0$ and $Z = \frac{S-\mu}{\sigma}$.

Central Limit Theorem: $Z \overset{(\approx)}{\sim} \mathrm{N}(0,1)$, so $F_S(s) \approx \Phi\left(s; \mu, \sigma^2\right)$.

Translated Gamma: $Z \overset{(\approx)}{\sim} \frac{Y-\alpha}{\sqrt{\alpha}}$ if $Y \sim$ gamma$(\alpha, 1)$ and $\gamma_Z = \gamma_Y$.
So $F_S(s) \approx G(s - x_0; \alpha, \beta)$ with G the gamma cdf and 3 fitted moments, i.e. $\alpha = \frac{4}{\gamma^2}$, $\beta = \frac{2}{\gamma\sigma}$, $x_0 = \mu - \frac{2\sigma}{\gamma}$.

Normal Power: $Z \overset{(\approx)}{\sim} U + \frac{1}{6}\gamma(U^2 - 1)$ with $U \sim \mathrm{N}(0,1)$, so

$$\begin{cases} \text{for quantiles:} & \Pr\left[Z \leq s + \frac{\gamma}{6}(s^2-1)\right] \approx p \text{ if } s = \Phi^{-1}(p); \\ \text{for cdf-values:} & \Pr[Z \leq z] \approx \Phi\left(\sqrt{\frac{9}{\gamma^2} + \frac{6z}{\gamma} + 1} - \frac{3}{\gamma}\right). \end{cases}$$

Collective model

Compound r.v. $S = X_1 + \cdots + X_N$ with $X_1, X_2, \ldots$ iid $\sim X$, independent of N.

Notation: $\mu_k = \mathrm{E}[X^k]$, $P(x) = \Pr[X \leq x]$, $F(s) = \Pr[S \leq s]$.

Then $\mathrm{E}[S] = \mathrm{E}[N]\mu_1$, $\mathrm{Var}[S] = \mathrm{E}[N]\mathrm{Var}[X] + \mathrm{Var}[N]\mu_1^2$, and $\mathrm{m}_S(t) = \mathrm{m}_N(\log \mathrm{m}_X(t))$.

$N \sim$ Poisson(λ): $\mathrm{E}[S] = \lambda\mu_1$, $\mathrm{Var}[S] = \lambda\mu_2$, $\mathrm{m}_S(t) = e^{\lambda[\mathrm{m}_X(t) - 1]}$.

Convolution formula: $f(x) = \sum_{n=0}^\infty p^{*n}(x)\Pr[N = n]$.

Number of claims $N \sim$ Poisson in a *Poisson process* (see below).

Parameter uncertainty: If $N|\Lambda = \lambda \sim$ Poisson(λ) and $\Lambda \sim$ gamma, then $N \sim$ NegBin.

Cumulation: If $M = L_1 + \cdots + L_N$ with $N \sim$ Poisson and L_i iid $\sim$ logarithmic(c) (so $\Pr[L = k] \propto \frac{c^k}{k}$, $k = 1, 2, \ldots$), then $M \sim$ NegBin.

Limit: For $\lambda = r(1-p)/p$ and large r, NegBin$(r, p) \approx$ Poisson(λ).

Properties of compound Poisson (cP) r.v.'s

If $S_j \sim \mathrm{cP}(\lambda_j, P_j)$ indep. and $\lambda_\Sigma = \sum \lambda_j$, then $\sum S_j \sim \mathrm{cP}\left(\lambda_\Sigma, \sum \lambda_j P_j/\lambda_\Sigma\right)$.

If $N_j \sim$ Poisson(λ_j) indep., then $\sum x_j N_j \sim \mathrm{cP}\left(\lambda_\Sigma, p(x_j) = \lambda_j/\lambda_\Sigma\right)$.

If $S = X_1 + \cdots + X_N \sim \mathrm{cP}(\lambda, p)$ and the frequency of amount x_j in S is $N_j = \#\{h | X_h = x_j\}$ (and therefore $S = \sum x_j N_j$), then $N_j \sim$ Poisson$(\lambda p(x_j))$ independent, $j = 1, 2, \ldots$

If X is integer, with $x_j = j\ \forall j$ (so $S = \sum j N_j$), the db of S can be computed by successively convolving the density of $1N_1 + 2N_2 + \cdots + (k-1)N_{k-1}$ with the (sparse) one of kN_k, $k = 2, 3, \ldots$ *(Sparse vector algorithm)*

Panjer's recursion If $N \sim \mathbf{Poisson}(\lambda)$ and $X \in \{0,1,\dots\} \sim p$, then $f(s) = \Pr[S=s]$ can be computed recursively:

starting value $f(0) = \mathrm{e}^{-\lambda(1-p(0))}$;

recursion formula $f(s) = \frac{1}{s}\sum_{h=1}^{s} \lambda h p(h) f(s-h)$, $s = 1,2,\dots$

If $N \sim \mathbf{Bin}(r,p)$ take $a = p/(1-p)$, $b = -a(r+1)$;
if $N \sim$ **NegBin**(r,p) take $a = 1-p$, $b = a(r-1)$.

Next: $f(0) = \begin{cases} \Pr[N=0] & \text{if } \ p(0)=0; \\ \mathrm{m}_N\big(\log p(0)\big) & \text{if } \ p(0)>0; \end{cases}$

$$f(s) = \frac{1}{1-ap(0)}\sum_{h=1}^{s}\left(a+\frac{bh}{s}\right)p(h)f(s-h),\ s=1,2,\dots$$

Stop-loss premiums: $\pi(k) = \pi(k-1) - [1-F(k-1)]$ with $\pi(0) = \mathrm{E}[N]\mathrm{E}[X]$, for integer X, k.

Approximations Use CLT, Translated Gamma and NP for compound r.v.'s with 'large' $\mathrm{E}[N]$ as well.

For $\mathrm{cP}(\lambda, X)$, use $\mu = \lambda\mathrm{E}[X]$, $\sigma^2 = \lambda\mathrm{E}[X^2]$, and $\gamma = \lambda\mathrm{E}[X^3]/\sigma^3$.

If cumulants κ_j of X are known, use as raw moments of X
$\mathrm{E}[X] = \kappa_1$, $\mathrm{E}[X^2] = \kappa_2 + \kappa_1^2$ and $\mathrm{E}[X^3] = \kappa_3 + 3\kappa_2\kappa_1 + \kappa_1^3$.

Fast Fourier Transform method to approximate compound db:
$(f_0, f_1, \dots, f_{n-1})' \approx \frac{1}{n}\mathbf{T}^{-} \mathrm{g}_N(\mathbf{T}^{+}\vec{p})$.

Here the matrices $\mathbf{T}^{\pm}$ are discrete Fourier transforms, n is so large that $\Pr[S \geq n] \approx 0$, and $g_N(z)$ is the pgf of N.

Collective model approximations

If the payment b_i in case of a claim on policy i is *fixed*, the individual model is $\widetilde{S} = \sum I_i b_i$, with $\Pr[I_i = 1] = q_i = 1 - \Pr[I_i = 0]$.

Approximate $F_{\widetilde{S}}$ by F_S with $S = \sum N_i b_i$ and $N_i \sim \mathrm{Poisson}(\lambda_i)$.

Taking $1 - \mathrm{e}^{-\lambda_i} = q_i \ \forall i$ gives an ***open*** collective model.

The collective model has $\lambda_i = q_i \ \forall i$.

Then $S \sim \mathrm{cP}(q_\Sigma, p(b_i) = q_i/q_\Sigma)$,
$S \sim \sum Z_i$ with $Z_i \sim \mathrm{cP}(1, I_i b_i)$,
$S \sim \mathrm{cP}\big(n, \frac{1}{n}\sum \Pr[I_i b_i \leq x]\big)$,
and $\mathrm{E}[S] = \mathrm{E}[\widetilde{S}]$ but $\mathrm{Var}[S] > \mathrm{Var}[\widetilde{S}]$.

Random claim amounts: replace $\widetilde{S} = \sum X_i$ by $S = \sum_i \sum_{j=1}^{N_i} X_i^{(j)}$ with $N_i \sim \mathrm{Poisson}(1)$, $X_i^{(j)} \sim X_i$, all independent.

Loss distributions For claim totals, consider $S = X_1 + \dots + X_N$ with $N \sim$ Poisson or (Neg)Bin, or some other counting db.

Generating samples from a distribution

1) Inversion method: if $U_i \sim$ Uniform(0,1) iid, then $F^{-1}(U_i)$, $i = 1,\dots,n$, is a sample from F.

2) Transform from another distribution, e.g. $-\log U \sim \exp(1)$.

3) Rejection method: draw a 'simpler' r.v. and accept it with the right probability.

Maximum likelihood estimation

Sample loglikelihood is $\ell(\alpha,\beta,\dots;\vec{y}) = \log\prod f_{Y_i}(y_i;\,\alpha,\beta,\dots)$.

To maximize ℓ, solve the ML-equations $\partial\ell/\partial\alpha = 0, \dots$ to get

1) analytical solutions (Poisson, Bin, (log-)normal, IG, Pareto);

2) partial solutions using which only a one-dimensional optimization is needed (NegBin, gamma);

3) no helpful solution: use a multidimensional optimization routine.

Mixture/combination of exponentials If $X, Y \sim \exp(1)$, $I \sim$ Bernoulli(γ), all independent, $0 < \alpha < \beta$ and $q = \frac{\beta\gamma}{\beta-\alpha}$, then $IX/\alpha + Y/\beta \sim p(x) = q\alpha\mathrm{e}^{-\alpha x} + (1-q)\beta\mathrm{e}^{-\beta x}$, $x > 0$.

If $0 \leq q \leq 1$ it is a *mixture*, if $q > 1$ a *combination* of exponentials.

Approximations for stop-loss premiums

Normal Power and CLT approximation for $\mathrm{E}[(S-d)_+]$:

Write $\mu = \mathrm{E}[S]$, $\sigma^2 = \mathrm{Var}[S]$, $\gamma = \gamma_S \geq 0$ and $Z = (S-\mu)/\sigma$.

Use that $\mathrm{E}[(S-d)_+] = \sigma\mathrm{E}[(Z-z)_+]$ if $z = (d-\mu)/\sigma$.

By NP, $\mathrm{E}[(Z-z)_+] \approx \varphi(w) + \frac{\gamma}{6}w\varphi(w) - z[1-\Phi(w)]$,

where $w = \sqrt{9/\gamma^2 + 6z/\gamma + 1} - 3/\gamma$ if $\gamma > 0$, $w = z$ if $\gamma = 0$.

Translated gamma approximation: if $S - x_0 \overset{(\approx)}{\sim} T \sim \mathrm{gamma}(\alpha,\beta)$, use $\mathrm{E}[(T-t)_+] = \frac{\alpha}{\beta}[1 - G(t;\alpha+1,\beta)] - t[1-G(t;\alpha,\beta)]$.

Rule of thumb: If $t > \mathrm{E}[U] = \mathrm{E}[W]$, then $\frac{\mathrm{E}[(U-t)_+]}{\mathrm{E}[(W-t)_+]} \approx \frac{\mathrm{Var}[U]}{\mathrm{Var}[W]}$.

Ruin theory

Poisson process $N(t)$ is a Poisson process if for all $t > 0$, $h > 0$ and each history $N(s)$, $s \leq t$, its increments satisfy:

$N(t+h) - N(t) \sim \mathrm{Poisson}(\lambda h)$.

The increments are *independent* and *stationary*; the process is *memoryless*.

The *infinitesimal increments* of a Poisson process have

$$\Pr[N(t+\mathrm{d}t) - N(t) = k \mid N(s), 0 \leq s \leq t] = \begin{cases} \lambda\mathrm{d}t & \text{if } k=0 \\ 1-\lambda\mathrm{d}t & \text{if } k=1 \\ 0 & \text{if } k=2,3,\dots \end{cases} + O((\mathrm{d}t)^2).$$

If T_j are times of an occurrence, then the *waiting times* $W_1 = T_1$, $W_j = T_j - T_{j-1}$, $j > 1$ are iid and exponential(λ).

Classical ruin process $U(t) = u + ct - S(t)$, $t \geq 0$, where

$U(t)$ = the insurer's random capital at time t;

$u = U(0)$ = the initial capital;

c = the (constant) premium income per unit of time;

$S(t) = X_1 + X_2 + \dots + X_{N(t)}$, with

$N(t)$ = number of claims till t, which is a Poisson process, and

X_i = the size of the i^{th} claim, assumed non-negative.

Ruin probability is $\psi(u) = \Pr[T < \infty]$,

where the *time of ruin* T in process $U(t)$ is the *defective* r.v.

$$T = \begin{cases} \min\{t \mid t \geq 0 \ \&\ U(t) < 0\}; \\ \infty \quad \text{if } U(t) \geq 0 \text{ for all } t. \end{cases}$$

Adjustment coefficient is the unique $R > 0$ solving $1 + (1+\theta)\mathrm{E}[X]R = \mathrm{m}_X(R)$.

Here θ is the *loading factor*, so $c = (1+\theta)\lambda\mathrm{E}[X]$.

If S is the total claims in a unit of time, then R also uniquely solves $\mathrm{e}^{Rc} = \mathrm{E}[\mathrm{e}^{RS}] \iff \mathrm{m}_{c-S}(-R) = 1 \iff c = \frac{1}{R}\log\mathrm{m}_S(R)$.

So if c is an exponential annual premium, the risk aversion is R.

Lundberg inequality $\psi(u) < \mathrm{e}^{-Ru}$. Always $\psi(0) = \frac{1}{1+\theta}$.

For *exponential*(β) claims, $R = \frac{\theta\beta}{1+\theta}$ and $\psi(u) = \frac{1}{1+\theta}\mathrm{e}^{-Ru}$.

Discrete time ruin model

Let $G_n = c - (S(n) - S(n-1))$ be the gain in year n, so $U(n) = u + G_1 + \dots + G_n$, $n = 1,2,\dots$

Define discrete ruin time $\widetilde{T} = \min\{n : U(n) < 0\}$, ruin probability $\widetilde{\psi}(u) = \Pr[\widetilde{T} < \infty]$, adjustment coefficient $\widetilde{R}$ solving $\mathrm{m}_G(-\widetilde{R}) = 1$.

In a Poisson process $\widetilde{R} = R$ but $\widetilde{\psi}(u) < \psi(u)\ \forall u$.

If $G_n \sim \mathrm{N}(\mu, \sigma^2)$, then $\widetilde{R} = 2\mu/\sigma^2$.

Lundberg inequality: $\widetilde{\psi}(u) < \mathrm{e}^{-\widetilde{R}u}$.

Risk measures

Value-at-Risk $\mathrm{VaR}[S;p] \overset{\text{def}}{=} F_S^{-1}(p)$.

Holding capital $d = \mathrm{VaR}[S; 1-i]$ leads to minimal value of cost plus expected overshoot $i \cdot d + \mathrm{E}[(S-d)_+]$.

VaR is not *subadditive*, since $\mathrm{VaR}[S+T;p] > \mathrm{VaR}[S;p] + \mathrm{VaR}[T;p]$ may hold.

Expected shortfall $\mathrm{ES}[S;p] \overset{\text{def}}{=} \mathrm{E}[(S - \mathrm{VaR}[S;p])_+]$.

Tail-Value-at-Risk $\mathrm{TVaR}[S;p] \overset{\text{def}}{=} \frac{1}{1-p}\int_p^1 \mathrm{VaR}[S;t]\,\mathrm{d}t$.

$\mathrm{TVaR}[S;p] = \mathrm{VaR}[S;p] + \frac{1}{1-p}\mathrm{ES}[S;p] = \inf_d\left\{d + \frac{1}{1-p}\pi_S(d)\right\}$.

TVaR *is* subadditive.

Conditional Tail Expectation $\mathrm{CTE}[S;p] \overset{\text{def}}{=} \mathrm{E}\big[S \mid S > \mathrm{VaR}[S;p]\big]$.

Aka Conditional Value-at-Risk (CVaR).

$\mathrm{CTE}[S;p] = \mathrm{TVaR}[S; F_S(F_S^{-1}(p))] \geq \mathrm{TVaR}[S;p] \geq \mathrm{VaR}[S;p]$.

TVaR $\equiv$ CTE for continuous r.v.'s only.

Bonus-malus systems

Loimaranta efficiency If $b(\lambda)$ is the *steady-state* premium for claim frequency λ, the efficiency is $e(\lambda) = \frac{\lambda}{b(\lambda)}\frac{\mathrm{d}b(\lambda)}{\mathrm{d}\lambda} = \frac{\mathrm{d}\log b(\lambda)}{\mathrm{d}\log\lambda}$.

So $b(\lambda(1+h)) \approx b(\lambda)(1+e(\lambda)h)$.