7.8.3 Normal and Student copulas

In this section we introduce two important families of copulas, the normal or Gaussian and the Student copulas.

Normal copula The multivariate normal copula by definition is the joint cdf of the ranks of a multinormal random vector. Starting from a vector of iid $\mathrm{N}(0,1)$ random variables $V_1,\ldots,V_n$, for any $n\times n$ matrix $\mathbf{A}$ we get a multinormal random vector $\vec{W}$ with mean vector $\mathrm{E}[\vec{W}]=\vec{\mu}$ and covariances $\Sigma_{ij}=\mathrm{Cov}[W_i,W_j]=\mathrm{E}[(W_i-\mu_i)(W_j-\mu_j)]=\left(\mathbf{AA}'\right)_{ij}$ by taking $\vec{W}=\vec{\mu}+\mathbf{A}\vec{V}$, see (7.62). It is easy to see that in the bivariate case, if we start from V_1,V_2 iid $\mathrm{N}(0,1)$, then (V_1, rV_1+sV_2) with $s=\sqrt{1-r^2}$ is standard bivariate normal with correlation r. This boils down to taking $\mathbf{A}=\left(\begin{smallmatrix}1&0\\r&s\end{smallmatrix}\right)$, such that $\mathbf{AA}'=\left(\begin{smallmatrix}1&r\\r&1\end{smallmatrix}\right)$.

If (X,Y) is bivariate normal, it has the bivariate normal copula. But the same holds for $(F^{-1}(\Phi(X;\mu_X,\sigma_X^2)),G^{-1}(\Phi(Y;\mu_Y,\sigma_Y^2)))$, for any choice of the marginal cdfs F and G. Then we speak of *meta-multinormal* r.v.'s.

In ex. 7.8.9 and 7.8.10, we prove that Blomqvist's $\beta=$ Kendall's $\tau=\frac{2}{\pi}\arcsin r$, but Spearman's $\rho=\frac{6}{\pi}\arcsin\frac{r}{2}$. The results also apply to pairs of components of a multinormal$(\vec{\mu},\Sigma)$ random vector. To be able to compute these various association measures, we need a lemma about the probability of a bivariate normal random pair landing in the upper quadrant.

Lemma 7.8.8 (Both bivariate normal components large)
If $(X,Y)\sim$ standard bivariate normal with correlation r, then

$$\Pr[X>0,Y>0]=\frac{1}{4}+\frac{1}{2\pi}\arcsin r. \tag{7.79}$$

Proof. The density of a standard bivariate normal distribution with correlation r is

$$\varphi(x,y;r)=\frac{1}{2\pi\sqrt{1-r^2}}\exp\left[-\frac{1}{2(1-r^2)}(x^2-2rxy+y^2)\right]. \tag{7.80}$$

By substituting $s=\frac{x-ry}{\sqrt{1-r^2}}$, so $x>0$ if and only if $s>\frac{-ry}{\sqrt{1-r^2}}$, and $\mathrm{d}x=\sqrt{1-r^2}\,\mathrm{d}s$, we get

$$\begin{aligned}\Pr[X>0,Y>0]&=\int_0^\infty\int_0^\infty\varphi(x,y;r)\,\mathrm{d}x\,\mathrm{d}y\\&=\frac{1}{2\pi}\int_0^\infty\int_{\frac{-ry}{\sqrt{1-r^2}}}^\infty \mathrm{e}^{-\frac{1}{2}(s^2+y^2)}\,\mathrm{d}s\,\mathrm{d}y.\end{aligned} \tag{7.81}$$

Now convert from Cartesian to polar coordinates by letting $s=u\cos\theta$, $y=u\sin\theta$. Then $s^2+y^2=u^2$, $\frac{y}{s}=\tan\theta$, and $\det\frac{\partial(s,y)}{\partial(u,\theta)}=u$, so $\mathrm{d}s\,\mathrm{d}y\to u\,\mathrm{d}u\,\mathrm{d}\theta$ in (7.81). We integrate over $u>0$, $0<\theta<\frac{\pi}{2}+\arctan\frac{r}{\sqrt{1-r^2}}=\frac{\pi}{2}+\arcsin r$. So

$$\begin{aligned}\Pr[X>0,Y>0] &= \frac{1}{2\pi}\int_0^{\pi/2+\arcsin r}\underbrace{\int_0^\infty u\mathrm{e}^{-u^2/2}\,\mathrm{d}u}_{=1}\,\mathrm{d}\theta \\ &= \frac{1}{4}+\frac{1}{2\pi}\arcsin r,\end{aligned} \tag{7.82}$$

as was to be proved. ∇

Student copula Suppose (X,Y) is a standard bivariate t distributed random pair with k degrees of freedom and correlation r. That means $(X,Y)\sim(U,V)/\sqrt{W/k}$, where (U,V) is standard bivariate normal with correlation coefficient r, independent of W which has a chi-square distribution with df $=k$. A multivariate Student distribution with standard marginals arises from random vectors $\vec{V}/\sqrt{W/k}$ with $\vec{V}$ standard multinormal. The corresponding copula is the cdf of its ranks. Conditionally on $W=w$, the pair (X,Y) has a bivariate normal distribution with zero means, variances k/w and correlation r. Therefore the joint density of (X,Y,W) is

$$\begin{aligned}f_{X,Y,W}(x,y,w) &= f_{X,Y|W}(x,y|w)f_W(w) \\ &= \frac{1}{2\pi\sqrt{1-r^2}}\sqrt{\frac{w}{k}}\exp\left(\frac{w}{k}\left(x^2-2rxy+y^2\right)\right) \\ &\quad\times\frac{1}{\Gamma(k/2)}\left(\frac{1}{2}\right)^{k/2}w^{k/2-1}\mathrm{e}^{-w/2}.\end{aligned} \tag{7.83}$$

The joint density of (X,Y) is $f(x,y)=\int f_{X,Y,W}(x,y,w)\,\mathrm{d}w$, leading to

$$\begin{aligned}f(x,y) &= \frac{1}{2\pi\sqrt{k}\sqrt{1-r^2}}\frac{1}{\Gamma(k/2)}\left(\frac{1}{2}\right)^{k/2} \\ &\quad\times\int_0^\infty w^{k/2-1/2}\mathrm{e}^{-\frac{1}{2}(1+(x^2-2rxy+y^2)/k)w}\,\mathrm{d}w \\ &= \frac{1}{2\pi\sqrt{1-r^2}}\left[1+\frac{1}{k(1-r^2)}(x^2-2rxy+y^2)\right]^{-(k+2)/2}\end{aligned} \tag{7.84}$$

for $-\infty<x,y<\infty$. The marginals of a bivariate t pair have univariate t-distributions with the same degrees of freedom, so for the marginal density f_X of X we also have

$$f_X(x)=\frac{\Gamma(\frac{k+1}{2})}{\Gamma(\frac{k}{2})\sqrt{k\pi}}\left(1+\frac{x^2}{k}\right)^{-(k+1)/2},\quad -\infty<x<\infty. \tag{7.85}$$

In equation (7.64) we gave the conditional distribution of one component of a bivariate normal pair, given the value of the other. In case of standard normal marginals, it can be written as

$$\frac{Y-rx}{\sqrt{1-r^2}}\,\Big|\,X=x \quad\sim\quad \mathrm{N}(0,1). \tag{7.86}$$

The proof of (7.64) is not given in this text, but it is in fact similar to the one for the corresponding result for multistudent random vectors given in the following lemma. It can also be inferred directly from it by letting $k \to \infty$.

Lemma 7.8.9 (Conditional distribution for bivariate Student)
If (X,Y) has a bivariate Student t_k distribution with standard marginals, k degrees of freedom and correlation r, then

$$\left(\frac{k+1}{k+x^2}\right)^{1/2} \frac{Y-rx}{\sqrt{1-r^2}} \,\Bigg|\, X=x \quad \sim \quad t_{k+1}. \tag{7.87}$$

Proof. Let us denote

$$\sigma(x) = \left[\frac{(1-r^2)(k+x^2)}{k+1}\right]^{-1/2}. \tag{7.88}$$

The lemma asserts that conditional on $X=x$, $\frac{Y-rx}{\sigma(x)}$ has the t-distribution with $k+1$ degrees of freedom, or equivalently,

$$\sigma(x) f_{Y|X}(rx+\sigma(x)y|x) = f_{t_{k+1}}(y), \quad -\infty < x, y < \infty, \tag{7.89}$$

where $f_{Y|X}(\cdot|x)$ denotes the conditional density of Y given $X=x$, and $f_{t_{k+1}}(\cdot)$ denotes the density of the t-distribution with $k+1$ degrees of freedom. We will directly verify the equality (7.89), using simple algebra and the fact that $x\Gamma(x) = \Gamma(x+1)$ for $x>0$. The proof of (7.89) proceeds as follows:

$$\begin{aligned}
\sigma(x) f_Y(rx+\sigma(x)y|X=x) &= \frac{\sigma(x) f(x, rx+\sigma(x)y)}{f_X(x)} \\
&= \frac{\sigma(x)\Gamma(\frac{k}{2})\sqrt{k\pi}}{2\pi\sqrt{1-r^2}\,\Gamma(\frac{k+1}{2})\left[1+\frac{x^2}{k}\right]^{-(k+1)/2}} \\
&\quad \times \left[1+\frac{1}{k(1-r^2)}\left(x^2-2rx(rx+\sigma(x)y)+(rx+\sigma(x)y)^2\right)\right]^{-(k/2+1)} \\
&= \frac{\Gamma(\frac{k}{2}+1)}{\Gamma(\frac{k+1}{2})\sqrt{\pi(k+1)}\left[1+\frac{x^2}{k}\right]^{-(k/2+1)}} \\
&\quad \times \left[1+\frac{1}{k(1-r^2)}\left(x^2-(rx)^2+\sigma^2(x)y^2\right)\right]^{-(k/2+1)} \\
&= \frac{\Gamma(\frac{k}{2}+1)}{\Gamma(\frac{k+1}{2})\sqrt{\pi(k+1)}}\left[\frac{k}{k+x^2}+\frac{x^2-(rx)^2+\sigma^2(x)y^2}{(k+x^2)(1-r^2)}\right]^{-(k/2+1)} \\
&= \frac{\Gamma(\frac{k}{2}+1)}{\Gamma(\frac{k+1}{2})\sqrt{\pi(k+1)}}\left[1+\frac{\sigma^2(x)y^2}{(k+x^2)(1-r^2)}\right]^{-(k/2+1)}.
\end{aligned} \tag{7.90}$$

Filling in $\sigma^2(x)$ as in (7.88), we finally get

$$\sigma(x) f_Y(rx + \sigma(x)y | X = x) = \frac{\Gamma(\frac{k}{2}+1)}{\Gamma(\frac{k+1}{2})\sqrt{\pi(k+1)}} \left[1 + \frac{y^2}{k+1}\right]^{-(k/2+1)} = f_{t_{k+1}}(y), \tag{7.91}$$

as claimed. ∇

A random vector $\vec{T} \sim \vec{\mu} + \vec{Z}\sqrt{k/W}$, with $W \sim \chi^2(k)$ independent of the multivariate normal$\left(\vec{0}, \Sigma\right)$ random vector $\vec{Z}$, has a *multivariate Student's* t_k distribution with parameters $\vec{\mu}, \Sigma, k$. The standard bivariate Student t_k distribution is a special case. The following results can be proved (see ex. 7.8.11):

- $\mathrm{E}[\vec{T}] = \vec{\mu}$ if $k > 1$ (if $k = 1$, the mean does not exist).
- $\vec{T}$ is symmetric, since $\vec{T} - \vec{\mu} \sim \vec{\mu} - \vec{T}$.
- For the i^{th} and j^{th} components, $r(T_i, T_j) = r(Z_i, Z_j)$ if $k > 2$.
- (Co-)variances are given by $\mathrm{Cov}[\vec{T}] = \frac{k}{k-2}\Sigma$ if $k > 2$.
- Blomqvist's β and Kendall's τ satisfy $\beta(T_i, T_j) = \tau(T_i, T_j) = \frac{2}{\pi} \arcsin r(Z_i, Z_j)$.
- Spearman's $\rho(T_i, T_j) = \frac{6}{\pi} \arcsin \frac{1}{2} r(Z_i, Z_j)$.

Elliptical distributions Multinormal and Student r.v.'s are special cases of elliptical random variables. Random n-vector $\vec{T}$ has an *elliptical distribution* if

$$\vec{T} \sim \vec{\mu} + \mathbf{A}\vec{U}R, \tag{7.92}$$

where the n-vector $\vec{\mu}$ is arbitrary, matrix $\mathbf{A}$ is $n \times k$, the random k-vector $\vec{U}$ is uniform on the unit sphere, and the random variable R is non-negative and independent of $\vec{U}$.

By symmetry,

$$\mathrm{Cov}[U_i, U_j] = \mathrm{Cov}[U_i, 1 - U_j], \quad i \neq j, \tag{7.93}$$

so for the covariance matrices, see also (7.62),

$$\mathrm{Cov}[\vec{U}] = \frac{1}{12}\mathbf{I}, \quad \mathrm{Cov}[\mathbf{A}\vec{U}] = \frac{1}{12}\mathbf{A}\mathbf{A}'. \tag{7.94}$$

If $\mathrm{E}[R^2] < \infty$, we get for the covariance and correlation matrices of $\vec{T}$:

$$\mathrm{Cov}[\vec{T}] = \mathrm{E}[R^2]\mathrm{Cov}[\mathbf{A}\vec{U}], \text{ so } \mathrm{Corr}[\vec{T}] = \mathrm{Corr}[\mathbf{A}\vec{U}]. \tag{7.95}$$

The support of $\vec{T}$, given $R = r$, is an ellipsoid, including its interior if $\mathbf{A}$ is not of full rank. In the bivariate case, each iso-density locus (the set of points all giving a particular value of the density) is an ellipse. If $R^2 \sim \chi^2(n)$, a multinormal random vector arises; dividing it by $\sqrt{W}$ with $W \sim \chi^2(k)$, or equivalently taking $R^2 \sim F(n,k)$, the Fisher distribution, gives multivariate Student's t_k random n-vectors.

If $\Pr[R = 0] = 0$, the same results as in ex. 7.8.11 for $\Pr[T_i > \mu_i, T_j > \mu_j]$ can be proved, and from that, also for their correlations r, β and τ. Examples exist, however, having $\rho(T_i, T_j) \neq \frac{6}{\pi} \arcsin \frac{r(T_i, T_j)}{2}$.

Another way to describe and handle elliptical distributions is by characteristic functions and so-called characteristic generators.

7.8.4 Tail dependence

Next to the correlation between random variables, another important property by which to determine which copula to choose is the *tail dependence*, measured by upper and lower *tail index*. The upper tail index is the probability of a huge X-value (of 'rank' $> u \uparrow 1$), given a huge Y-value. If it is zero, simultaneous disasters have low probability.

Definition 7.8.10 (Lower and upper tail index)
For a pair (X,Y) with ranks (U,V) and copula $C(\cdot,\cdot)$, the *tail indices* are defined as

$$\begin{aligned}\text{Upper tail index: } \lambda^{\text{upp}} &\stackrel{\text{def}}{=} \lim_{u\uparrow 1} \Pr[U > u \,|\, V > u] = \lim_{v\uparrow 1} \frac{1-2u+C(u,u)}{1-u};\\ \text{Lower tail index: } \lambda^{\text{low}} &\stackrel{\text{def}}{=} \lim_{u\downarrow 0} \Pr[U \le u \,|\, V \le u] = \lim_{u\downarrow 0} \frac{C(u,u)}{u}.\end{aligned} \tag{7.96}$$

∇

It is easy to check that for the independence copula $C(u,v) = uv$ as well as the FGM-copula, both tail indices are zero. For the comonotone copula $C(u,v) = \min(u,v)$, they are 1, and for a mixed copula $C(u,v) = p\min(u,v) + (1-p)uv$, they are p.

To derive the tail indices for the normal and Student copulas, we prove:

Lemma 7.8.11 (Tail indices and derivative of copula)
If random pair (X,Y) has ranks (U,V), its lower tail index is given by

$$\lambda^{\text{low}} = \lim_{u\downarrow 0} \{\Pr[U \le u \,|\, V = u] + \Pr[V \le u \,|\, U = u]\}. \tag{7.97}$$

Analogous for the upper tail index.

Proof. By l'Hôpital's rule, we have

$$\lambda^{\text{low}} = \lim_{u\downarrow 0} \frac{C(u,u)}{u} = \lim_{u\downarrow 0} \frac{\mathrm{d}C(u,u)}{\mathrm{d}u}, \tag{7.98}$$

and

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}u}C(u,u) &= (C(u+\mathrm{d}u, u+\mathrm{d}u) - C(u,u))/\mathrm{d}u\\ &= (\Pr[U \le u+\mathrm{d}u, V \in (u, u+\mathrm{d}u)]\\ &\quad + \Pr[U \in (u, u+\mathrm{d}u), V \le u])/\mathrm{d}u\\ &= \Pr[U \le u \,|\, V = u] f_V(u) + \Pr[V \le u \,|\, U = u] f_U(u)\\ &= \Pr[U \le u \,|\, V = u] + \Pr[V \le u \,|\, U = u]\end{aligned} \tag{7.99}$$

since $f_U(u) = f_V(u) = 1$. ∇

For a normal copula, the ranks of X and Y are exchangeable by symmetry, so if $r < 1$, using (7.86) we get by substituting $x = \Phi^{-1}(u)$ in (7.97):

$$\begin{aligned}\lambda^{\text{low}} &= \lim_{u\downarrow 0}\{\Pr[U \le u \,|\, V = u] + \Pr[V \le u \,|\, U = u]\} \\ &= \lim_{x\to-\infty} 2\Pr[X \le x \,|\, Y = x] \\ &= \lim_{x\to-\infty} 2\Phi\left(\frac{x - rx}{\sqrt{1-r^2}}\right) = 0.\end{aligned} \tag{7.100}$$

By symmetry, the upper tail index λ^{upp} is zero as well.

Now consider a Student(df=k) copula with correlation r. By Lemma 7.8.9, the tail indices are given by

$$\begin{aligned}\lambda^{\text{low}} = \lambda^{\text{upp}} &= \lim_{x\to-\infty} 2F_{t_{k+1}}\left(\sqrt{\frac{k+1}{k+x^2}}\,\frac{x - rx}{\sqrt{1-r^2}}\right) \\ &= 2\left\{1 - F_{t_{k+1}}\left(\sqrt{(k+1)(1-r)/(1+r)}\,\right)\right\}.\end{aligned} \tag{7.101}$$

Note that even for negative correlation, the probability of joint disasters is positive.

7.9 Exercises

Section 7.8, subsections 3–4

7. Show that Spearman's $\rho(X,Y)$ is the normed concordance probability with $(X^\perp, Y^\perp)$, where $X^\perp \sim X$, $Y^\perp \sim Y$ are independent of (X,Y) and of each other.
8. Let $(X,Y) \sim$ bivariate normal with $r(X,Y) = r$. Prove $\Pr[X > \mu_X, Y > \mu_Y] = \frac{1}{4} + \frac{1}{2\pi}\arcsin r$.
9. Prove that if (X,Y) is bivariate normal, then Blomqvist's β, Kendall's τ and Pearson's r are related as $\beta = \tau = \frac{2}{\pi}\arcsin r$.
10. Prove that if (X,Y) is bivariate normal, then Spearman's $\rho = \frac{6}{\pi}\arcsin\frac{r}{2}$.
11. Consider $\vec{T} \sim \vec{\mu} + \vec{Z}\sqrt{k/W}$, with $W \sim \chi^2(k)$ independent of the multinormal$(\vec{0}, \Sigma)$ random vector $\vec{Z}$. So $\vec{T}$ has a multivariate Student's t_k distribution. For the ith and jth components of $\vec{Z}$, write $r = r(Z_i, Z_j)$. Show that $r(T_i, T_j) = r$ if $k > 2$, $\Pr[T_i > \mu_i, T_j > \mu_j] = \frac{1}{4} + \frac{1}{2\pi}\arcsin r$, and $\beta(T_i, T_j) = \tau(T_i, T_j) = \frac{2}{\pi}\arcsin r$.

Section 7.8, subsections 3–4 — Hints (see Appendix B)

7. Prove $\rho(X,Y) = 12(\mathrm{E}[UV] - \frac{1}{4})$, $\mathrm{E}[UV] = \mathrm{E}[(1-U)(1-V)] = \Pr[S < U, T < V]$ with U,V,S,T the ranks with $X,Y,X^\perp,Y^\perp$.
8. Trivial from Lemma 7.8.8.
9. $\beta = 2\times\Pr[(X-\mu_X)(Y-\mu_Y) > 0] - 1$ and $(X-\mu_X, Y-\mu_Y) \sim \cdots$;
   $\tau = 2\times\Pr[(X-X^\bullet)(Y-Y^\bullet) > 0] - 1$.
10. $\rho = 12\times\Pr[(X-X^\perp)(Y-Y^\perp) > 0] - 3$.
11. Write $S = \sqrt{k/W}$. then $\mathrm{E}[(Z_iS)(Z_jS)] = \mathrm{E}[Z_iZ_j]\mathrm{E}[S^2]$.