A Robust Rational Route to Randomness in a Simple Asset Pricing Model^{*}

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Abstract

We investigate asset pricing dynamics in an adaptive evolutionary asset pricing model with fundamentalists, trend followers and a market maker. Agents can choose between a fundamentalist strategy at positive information cost or choose a trend following strategy for free. Price adjustment is proportional to the excess demand in the asset market. Agents asynchronously update their strategy according to realized net profits in the recent past. As agents become more sensitive to differences in strategy performance, the fundamental steady state becomes unstable and multiple steady states may arise. As the traders' sensitivity to differences in fitness increases, a bifurcation route to chaos sets in due to homoclinic bifurcations of stable and unstable manifolds of the fundamental steady state.

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1 Introduction

Agent based evolutionary modeling of financial markets is becoming increasingly popular. Recent work on heterogeneous agent modelling includes, for example, the computational oriented work on the Santa Fe artificial stock market by Arthur et al. (1997) and LeBaron et al. (1999), the stochastic multi-agent models of Lux and Marchesi (1999, 2000) and the evolutionary markets based on out-of-equilibrium price formation rules by Farmer and Joshi (2002). An early example of a heterogeneous agents financial market model is Zeeman (1974); other examples include Frankel and Froot (1988), Kirman (1991), Chiarella (1992), Brock (1993, 1997), Lux (1995), Gaunersdorfer (2000) and Westerhoff (2003). See e.g. LeBaron (2000) for a good recent survey of the literature.

Much of the interacting agents work is computationally oriented and the simulations suggest a rich possibility of complicated asset price fluctuations ranging from stable close to the fundamental price fluctuations to persistent deviations from the fundamental and irregular, seemingly unpredictable asset price fluctuations. Brock and Hommes (1997) have proposed a simple evolutionary framework for endogenous strategy selection, based upon the well known discrete choice model. The key feature of these systems is that strategies that have performed well, as measured e.g. by realized profits in the recent past, attract more followers. Brock and Hommes (1998), henceforth BH, apply this evolutionary framework to a standard asset pricing equilibrium model where investors can either buy a risk free asset paying a fixed return or invest in a risky asset paying an uncertain dividend. One of their main findings is that as the intensity of choice, measuring the sensitivity of traders to differences in strategy performance, increases the fundamental steady state becomes unstable and bifurcations routes to periodic and even chaotic asset price fluctuations occur with irregular switching between close to the fundamental steady state price fluctuations and temporary speculative bubbles. BH consider an equilibrium asset pricing model, where in each period a Walrasian auctioneer sets the market equilibrium price such that total demand equals total supply.

An important question is how stability is related to the market institution. In this paper we study a simple 2-type example with fundamentalists versus trend followers in a different market institution. Instead of a Walrasian market equilibrium price scenario, we consider a *market maker scenario*, with a simple price adjustment process where in each period the price change is proportional to the excess demand in the market. The remaining stocks that are in excess demand (supply) will be supplied out of (added to) the inventory of a market maker. Financial market heterogeneous agent models with market makers have also been considered e.g. by Day and Huang (1994), Lux (1995), Chiarella et al. (2002) and Chiarella and He (2002).

A second important extension is that we allow for *asynchronous* updating of beliefs or strategies, whereas BH considered *synchronous* updating of beliefs with all agents updating beliefs in each period. BH have shown that under synchronous updating, complicated chaotic asset price fluctuations can arise. To illustrate the importance of the role of synchronous/asynchronous updating of beliefs, let us recall a classic discussion of the results of Nowak and May (1992). The paper of Nowak and May (1992) presents numerical simulations of prisoner's dilemma games in a spatial context and shows that chaotic behaviour occurs. However, Huberman and Glance (1993) stressed that if the cells on the lattice were updated not synchronously as usually in cellular automata, but asynchronously –picking a player randomly and replacing it with the best player in a neighbourhood– chaos disappears and the evolution leads to victory of the defectors and no survival of the cooperators. Similar results were obtained in Mukherij et al. (1995). Nowak et al. (1994) however showed that this effect was only valid for a limited part of the parameter space. This discussion shows the importance of investigating evolutionary asset pricing models with asynchronous updating of beliefs. In our model updating of beliefs is asynchronous, since only a (fixed) fraction of the community of traders updates their strategies in each period.

A third difference with BH is that we consider the possibility of a positive supply of outside shares of the risky asset. This means that, in equilibrium traders require a positive risk premium to hold all risky shares. The risk premium affects the fitness measure, and thus affects the evolutionary updating of fractions. We investigate in particular how this risk premium affects the stability of the fundamental steady state and global asset pricing fluctuations.

Despite these important differences compared to the BH-model, the global picture of asset price dynamics is surprisingly similar as in BH. In particular, we show that a *rational route to randomness*, that is, a bifurcation route to strange attractors occurs when the intensity of choice to switch strategies increases. Our results show that many global dynamic features are robust with respect to details of modelling market institution and evolutionary strategy switching. The mathematical mechanism explaining this bifurcation route is a homoclinic bifurcation between the stable and unstable manifold of the saddle point fundamental steady state. Due to our simple price adjustment rule the model reduces to a 2-D system, and the homoclinic bifurcation between the stable and unstable manifolds of the fundamental steady state can be nicely illustrated in the 2-D phase space, and becomes intuitively plausible. Other economic applications of homoclinic bifurcation theory include De Vilder (1996), Pintus, Sands and De Vilder (2000), Goeree and Hommes (2000), Yokoo (2000) and Droste, Hommes and Tuinstra (2002).

The paper is organized as follows. Section 2 presents the model. In section 3 we investigate local stability and bifurcations of the fundamental steady state. Section 4 focusses on global dynamics and section 5 concludes. All proofs are contained in an appendix.

2 The Model

The initial part of the presentation of the model follows the asset pricing equilibrium model with heterogeneous beliefs in BH, but then deviates by replacing the Walrasian market clearing equilibrium price scenario by a market maker scenario as used in Chiarella et al.(2002) and Chiarella and He (2002); see also Brock (1997) and Hommes (2001) for more extensive discussions of the equilibrium model.

Agents can invest in a risk free asset or in a risky asset. The risk-free asset is perfectly elastically supplied at a gross return R > 1. p_t denotes the price (ex-dividend) of the risky asset and y_t the (stochastic) dividend process. Tomorrow's wealth of trader type h is described by

$$W_{h,t+1} = RW_{ht} + (p_{t+1} + y_{t+1} - Rp_t)z_{ht},$$
(1)

where z_{ht} is the number of shares of the risky asset purchased at time t by trader type h. Let E_{ht} , V_{ht} be the beliefs or forecasts of trader type h about the conditional expectation and conditional variance of tomorrow's wealth, based upon a publicly available information set consisting of past prices and dividends. Assuming that the investors are myopic mean-variance maximizer, the demand for shares z_{ht} solves

$$Max_{z_{ht}} \{ E_{ht}(W_{h,t+1}) - \frac{a}{2} V_{ht}(W_{h,t+1}) \},$$
(2)

where a > 0 denotes the risk aversion assumed to be equal and constant for all investors. The demand z_{ht} for risky assets by trader type h is then given by

$$z_{ht} = \frac{E_{ht}(p_{t+1} + y_{t+1} - Rp_t)}{aV_{ht}(p_{t+1} + y_{t+1} - Rp_t)} = \frac{E_{ht}(p_{t+1} + y_{t+1} - Rp_t)}{a\sigma^2},$$
(3)

where we have assumed that beliefs about conditional variances of the excess returns are the same for all investors and constant over time, i.e. $V_{ht} \equiv \sigma^2$.

At this point, let us briefly discuss a world where all investors are identical and expectations are homogeneous. Equilibrium of demand and supply implies

$$\frac{E_{ht}(p_{t+1} + y_{t+1} - Rp_t)}{a\sigma^2} = z_s,$$
(4)

where z_s denotes the supply of shares per investor in the market, assumed to be fixed and constant over time. The market equilibrium equation can be rewritten as

$$Rp_t = E_t(p_{t+1} + y_{t+1}) - a\sigma^2 z_s.$$
(5)

The quantity $a\sigma^2 z_s$ may be interpreted as the risk premium investors require for holding all risky assets. It is well known that using the market equilibrium equation repeatedly and assuming that the transversality condition

$$\lim_{t \to \infty} \frac{E_t(p_{t+k})}{R^k} = 0 \tag{6}$$

holds, the price of the risky asset is uniquely determined by

$$p_t^* = \sum_{k=1}^{\infty} \frac{E_t(y_{t+k}) - a\sigma^2 z_s}{R^k}.$$
(7)

This price p_t^* is called the fundamental rational expectations (RE) price or the fundamental price for short. The fundamental price is completely determined by economic fundamentals and given by the discounted sum of expected future dividends adjusted by the risk premium. Assuming that dividends are independently and identically distributed (IID) with constant mean value $E(y_t) = \bar{y}$, the fundamental price is constant and given by

$$p^* = \sum_{k=1}^{\infty} \frac{\bar{y} - a\sigma^2 z_s}{R^k} = \frac{\bar{y} - a\sigma^2 z_s}{R - 1}.$$
(8)

As in Chiarella et al.(2002) and Chiarella and He (2002), we assume that there are three classes of traders in the asset market: two groups of speculators - fundamentalists and trend followers - and a market maker. Both fundamentalists and trend followers have correct expectations on dividends, and thus forecast $E_{ht}(y_{t+1}) = E_t(y_t) = \bar{y}$. Expectations on future prices of fundamentalists and trend followers are described by

$$E_{1t}(p_{t+1}) = E_t(p_{t+1}^*) = p^*, (9)$$

$$E_{2t}(p_{t+1}) = E_t(p_{t+1}^*) + g(p_t - p_t^*) = p^* + g(p_t - p^*), \quad g > 0.$$
⁽¹⁰⁾

At the beginning of period t, after observing the price p_t , the demand z_{ht} (h = 1, 2) for the risky asset by fundamentalists and trend followers is obtained by substituting the common dividend forecast \bar{y} and the forecasts (9) and (10) into the general demand function (3). Aggregate excess demand in period t is then given by

$$z_{e,t} = \sum_{h=1}^{2} n_{ht} z_{ht} - z_s, \tag{11}$$

where n_{ht} is the fraction of investors of type h = 1, 2. The role of the market maker is to provide liquidity. When excess demand is positive, the market maker sells shares out of his inventory at the going market price to clear the market; when excess demand is negative, the market maker buys those shares at the going market price and adds them to his inventory. The market maker buys from the investors when prices are high and sell when prices are low. The investors, both fundamentalists and technical analysts, are assumed to pay a fee to the market maker for providing liquidity and maintaining the market institution. At the end of period t, after all transactions have been carried out, the market price adjusts proportional to the observed excess demand. To be precise,

$$p_{t+1} = p_t + \mu z_{e,t} \tag{12}$$

with the parameter $\mu > 0$ denoting the correspondiong speed of adjustment. This stylized price adjustment rule captures the key feature of "the law of demand and supply", namely that prices rise when there is excess demand and prices fall when there is excess demand. The same price adjustment rule has been used recently by Chiarella et al.(2002) and Chiarella and He (2002), who argue that it provides a stylized behavioural rule of the market maker's role of market-clearing and price-impact. Day and Huang (1990), Lux (1995), Lux and Marchesi (2000) and Farmer and Joshi (2002) have used similar price adjustment rules in financial market models, where price changes are based upon excess demand or upon *changes* in (excess) demand. A similar price adjustment rule is also used in the well-known tâtonnement process, in continuous time e.g. by Arrow et al.(1959), Scarf (1960) and Arrow and Hahn (1971), and in discrete time e.g. by Saari (1985), Bala and Majumdar (1992), Day and Pianigiani (1991), Weddepohl (1995), Goeree et al. (1998) and Tuinstra (1999, 2000). In our asset pricing setting, an advantage of the simple price adjustment rule (12) is that the model remains analytically tractible and reduces to a 2-dimensional system.

Finally, the evolutionary part of the model describes how beliefs are updated over time, that is, how the fractions n_{ht} evolve over time. Traders are boundedly rational and tend to choose the strategy that has performed well in the recent past. The updated fractions are formed on the basis of discrete choice probabilities, that is,

$$n_{h,t+1} = \alpha n_{ht} + (1-\alpha) \frac{e^{\beta(U_{ht} - C_h)}}{Z_t}, \qquad Z_t = \sum_{h=1}^2 e^{\beta(U_{ht} - C_h)}.$$
(13)

The parameter α , $0 \le \alpha \le 1$ is the fraction of traders that sticks to its previous strategy, whereas a fraction $1 - \alpha$ updates their strategy according to their fitness U_{ht} . In the extreme case $\alpha = 1$, agents never update their beliefs; the other extreme case $\alpha = 0$ corresponds to synchronous updating of beliefs, where all agents update beliefs in each period. The general case $0 < \alpha < 1$ corresponds to asynchronous updating of beliefs. The parameter $\beta \ge 0$ is the intensity of choice measuring how sensitive agents are with respect to differences in the fitness of the prediction strategies. In the extreme case $\beta = 0$ the traders are insensitive to differences in fitness and both fractions converge to 1/2. In the other extreme case, $\beta = \infty$, in each period a fraction $1 - \alpha$ of the traders switches to the strategy that has highest fitness. $C_h \ge 0$ represents the average costs per period incurred by the investors of type h. The general idea is that more sophisticated strategies require higher costs, e.g. due to information gathering. In what follows we will assume that the costs for fundamentalists are higher than the costs for trend followers. Finally, U_{ht} is the fitness measure of trading strategy h, which is defined as a weighted average of the realized profits of trader type h, that is,

$$U_{ht} = (p_{t+1} + y_{t+1} - Rp_t)z_{ht} + wU_{h,t-1},$$
(14)

where w is a weight parameter. In general, evolutionary fitness is thus a weighted average of realized profits with exponentially declining weights. For analytical tractability we will focus on the case w = 0, where fitness is the most recently observed realized profit.

Let us now summarize the complete evolutionary adaptive belief system:

$$p_{t+1} = p_t + \mu(n_{1t}z_{1t} + n_{2t}z_{2t} - z_s), \tag{15}$$

$$n_{1,t+1} = \alpha n_{1t} + (1-\alpha) \frac{e^{\beta((p_{t+1}+y_{t+1}-Rp_t)z_{1t}-C_1)}}{Z_t},$$
(16)

$$n_{2,t+1} = \alpha n_{2t} + (1-\alpha) \frac{e^{\beta((p_{t+1}+y_{t+1}-Rp_t)z_{2t}-C_2)}}{Z_t},$$
(17)

where

$$z_{1t} = \frac{p^* + \bar{y} - Rp_t}{a\sigma^2},\tag{18}$$

$$z_{2t} = \frac{p^* + g(p_t - p^*) + \bar{y} - Rp_t}{a\sigma^2},$$
(19)

$$Z_t = e^{\beta((p_{t+1}+y_{t+1}-Rp_t)z_{1t}-C_1)} + e^{\beta((p_{t+1}+y_{t+1}-Rp_t)z_{2t}-C_2)}.$$
(20)

The main differences with the adaptive belief systems of BH are thus that we have a market maker scenario with a price adjustment rule based upon excess demand, instead of a Walrasian equilibrium, and we have asynchronous updating of strategies instead of synchronous updating. In what follows it will be convenient to work in deviations from the benchmark fundamental price. Let

$$x_t = p_t - p^*, \tag{21}$$

that is, x_t is the *deviation* of the price from its fundamental value. Furthermore let

$$m_t = n_{1t} - n_{2t}, (22)$$

that is, m_t is the difference in fractions. A straightforward computation shows that the evolutionary adaptive belief system can be written as the dynamic system

$$x_{t+1} = \left(1 - \frac{\mu R}{a\sigma^2} + \frac{\mu g}{2a\sigma^2}(1 - m_t)\right)x_t,$$
(23)

$$m_{t+1} = \alpha m_t + (1 - \alpha) \tanh\left[\frac{\beta}{2} \left(\frac{g}{a\sigma^2} \left(R - \left(1 - \frac{\mu R}{a\sigma^2} + \frac{\mu g}{2a\sigma^2} (1 - m_t)\right)\right) x_t^2 - g z_s x_t - C - \frac{g}{a\sigma^2} x_t \epsilon_{t+1}\right)\right],$$
(24)

with $\epsilon_{t+1} = y_{t+1} - \bar{y}$ the noise term on dividends and $C = C_1 - C_2 > 0$, that is, per period information gathering costs for fundamentalists are higher than for chartists. Notice that the supply of outside shares z_s affects the fundamental price p^* in (8) and therefore shows up in the realizations of profits of trader types, and thus affects the difference in fractions in (24). We are thus able to study the effect of the risk premium upon the evolutionary dynamics by considering positive supply of outside shares, $z_s > 0$.

We will refer to the case without noise, i.e. $\epsilon_{t+1} \equiv 0$, as the *deterministic skeleton* and study its stability and bifurcations in detail.¹ From a mathematical viewpoint, the deterministic skeleton is a 2-dimensional (2-D) nonlinear dynamical system given by

$$\mathcal{F}: \begin{pmatrix} x_t \\ m_t \end{pmatrix} \longrightarrow \begin{pmatrix} x_{t+1} \\ m_{t+1} \end{pmatrix} = \begin{pmatrix} F_1(x_t, m_t) \\ F_2(x_t, m_t) \end{pmatrix}$$
(25)

where

$$F_1(x,m) = v(m)x \tag{26}$$

with

$$v(m) = \left(1 - \frac{\mu R}{a\sigma^2} + \frac{\mu g}{2a\sigma^2}\right) - \frac{\mu g}{2a\sigma^2}m,\tag{27}$$

and

$$F_2(x,m) = \alpha m + (1-\alpha) \tanh[\beta w(x,m)]$$
(28)

with

$$w(x,m) = \frac{1}{2} \left(\frac{g}{a\sigma^2} (R - v(m)) x^2 - g z_s x - C \right).$$
⁽²⁹⁾

¹In order to understand the properties of a stochastic model, it is important to understand the properties of the deterministic skeleton; see e.g. Tong (1990). The effect of the stochastic factors on the evolutionary competition will be investigated in future work.

3 Local Stability and Bifurcations

In this section we investigate the local stability and bifurcations of steady states for $0 \le \alpha < 1$, i.e. we exclude the case $\alpha = 1$, where fractions never change. Without loss of generality, we can normalize $a\sigma^2 = 1.^2$ To simplify the discussion and presentation of the results below, we will focus on gross return of the risk-free asset in the plausible range 1 < R < 1.2.

It should be clear that for any $\beta \ge 0$, $E = (0, -\tanh(\frac{\beta C}{2}))$ is a steady state of the system. Obviously when x = 0 (or equivalently $p = p^*$), the price is at its benchmark fundamental value, and we will therefore refer to E as the fundamental steady state. Notice that when C > 0, E moves from (0,0) to (0,-1) along the m-axes as β goes from 0 to ∞ . This is due to the fact that at the fundamental steady state both the fundamental and the trend following strategy yield the same forecast. There is no point in paying any cost in a steady state for a trading strategy that yields no extra profit, so the fraction of the cheap trend following strategy increases as the intensity of choice β increases. In order to compare our results directly to the Walrasian senario of BH, we first study the stability of the fundamental steady state E when the supply of outside shares is zero, i.e. when $z_s = 0$. All proofs of propositions are given in the Appendix.

Proposition 1. (Stability & bifurcations of fundamental steady state for $z_s = 0$; see figure 1) Let $\beta_{pd} = \frac{2}{C} \tanh^{-1}(\frac{2R}{g} - 1 - \frac{4}{\mu g}), \beta^* = \frac{2}{C} \tanh^{-1}(\frac{2R}{g} - 1)$ and let $E = (0, -\tanh(\frac{\beta C}{2}))$ be the fundamental steady state. We have:

- 1. For 0 < g < R (see Figure 1a):
 - (i) $0 < \mu < \frac{2}{B}$: E is globally stable for all $\beta \ge 0$;
 - (ii) $\frac{2}{R} < \mu < \frac{4}{2R-g}$: E is locally stable, but not globally stable for all $\beta > 0$ (for $\mu = \frac{2}{R}$ a two-cycle is born at infinity);
 - (iii) $\frac{4}{2R-g} < \mu < \frac{2}{R-g}$: E is unstable for $0 \le \beta < \beta_{pd}$ and locally stable for $\beta > \beta_{pd}$; E undergoes a (subcritical) period-doubling bifurcation at $\beta = \beta_{pd}$;
 - (iv) $\mu > \frac{2}{B-a}$: E is unstable for all $\beta \ge 0$.
- 2. For R < g < 2R (see Figure 1b):
 - (i) $0 < \mu < \frac{2}{R}$: E is globally stable for $0 \le \beta < \beta^*$ and unstable for $\beta > \beta^*$; E undergoes a (supercritical) pitchfork bifurcation at $\beta = \beta^*$;
 - (ii) $\frac{2}{R} < \mu < \frac{4}{2R-g}$: E is locally stable for $0 < \beta < \beta^*$ and unstable for $\beta > \beta^*$; E undergoes a (supercritical) pitchfork bifurcation at $\beta = \beta^*$;
 - (iii) $\mu > \frac{4}{2R-g}$: E is unstable for $0 \le \beta < \beta_{pd}$, locally stable for $\beta_{pd} < \beta < \beta^*$, and unstable for $\beta > \beta^*$; E undergoes a (subcritical) period-doubling bifurcation at $\beta = \beta_{pd}$ and a (supercritical) pitchfork bifurcation at $\beta = \beta^*$.
- 3. For g > 2R:

E is unstable for all $\mu > 0$ and $\beta \ge 0$.



Figure 1: Stability and bifurcations of the fundamental steady state in the (β, μ) plane. The curve indicated by β_{pd} is the subcritical period doubling bifurcation curve. The horizontal line at $\mu = 2/R$ corresponds to the creation of a 2-cycle at infinity. The vertical line $\beta = \beta^*$ corresponds to the supercritical pitchfork bifurcation in which two stable non-fundamental steady states are created.

The stability of the fundamental steady state is determined jointly by the price adjust speed μ , the intensity of choice β and the trend extrapolation parameter g. We compare the above result with Lemma 2 in BH (p.1249) under the Walrasian equilibrium scenario. Under the Walrasian scenario, in the case of weak trend extrapolation (0 < g < R) the fundamental steady state is always globally stable. Under the market maker scenario the situation is more complex and depends upon the speed of adjustment μ . For small price adjustment speed $(0 < \mu < \frac{2}{R})$ the fundamental steady state is globally stable. For larger adjustment speed $\mu > \frac{2}{R}$, the fundamental steady state is not globally stable, due to the fact that there exists a 2-cycle, which is created at infinity for $\mu = \frac{2}{R}$. This feature may be seen as an artifact of the price adjustment rule. For large adjustment speeds strong overshooting can occur leading to exploding, large amplitude up and down price fluctuations and an (unstable) 2-cycle far away from the fundamental steady state. From now on, we will therefore mainly focus on the case of 'small' price adjustment speed $(0 < \mu < \frac{2}{R})$ and ignore this extreme form of overshooting.

In the case of very strong trend extrapolation (g > 2R), both for the Walrasian scenario and the market maker scenario, the fundamental steady state is always unstable for all values of the adjust speed of price μ and the intensity of choice β . The most interesting case occurs for a strong trend extrapolation parameter (R < g < 2R): in both the Walrasian scenario and the market maker scenario the fundamental steady state becomes unstable through a supercritical pitchfork bifurcation. The following result shows that two non-fundamental steady states are created in this pitchfork bifurcation and discusses their stability:

Proposition 2. (Stability & bifurcation of non-fundamental steady states for $z_s = 0$; see Figure 2a)

For
$$R < g < 2R$$
 and $0 < \mu < \frac{2}{R}$, let $x^* = \frac{\sqrt{C - \frac{2}{\beta} \tanh^{-1}(\frac{2R}{g} - 1)}}{\sqrt{g(R-1)}}$, $\beta_{NS} = \frac{2}{C} \tanh^{-1}(\frac{2R}{g} - 1) + \frac{2R}{\beta} \ln^{-1}(\frac{2R}{g} - 1)$

²For the general case, one may use the transformation $\frac{\mu}{a\sigma^2} \mapsto \tilde{\mu}, \frac{\beta}{a\sigma^2} \mapsto \tilde{\beta}, a\sigma^2 z_s \mapsto \tilde{z}_s, a\sigma^2 C \mapsto \tilde{C}$, and then in the transformed system we have $a\sigma^2 = 1$.

 $\frac{g(R-1)}{\mu R(g-R)(2R-1)C}.$ We have:

- 1. For $0 \leq \beta < \beta^*$ the fundamental steady state $E = (0, -\tanh(\frac{\beta C}{2}))$ is the unique steady state;
- 2. For $\beta > \beta^*$ there exist two non-fundamental steady states $E_r = (x_r, 1 \frac{2R}{g})$ with $x_r = x^*$ and $E_l = (x_l, 1 \frac{2R}{g})$ with $x_l = -x^*$; moreover
 - (i) E_r and E_l are locally stable for $\beta^* < \beta < \beta_{NS}$ and unstable for $\beta > \beta_{NS}$;
 - (ii) both E_r and E_l undergo a supercritical Neimark-Sacker bifurcation at $\beta = \beta_{NS}$ for $\alpha < 0.6$.

Immediately after the supercritical pitchfork bifurcation the two non-fundamental steady states E_r and E_l are locally stable. As the intensity of choice further increases these nonfundamental steady states become unstable due to a supercritical Neimark-Sacker bifurcation in which two attracting invariant circles are created with periodic or quasi-periodic dynamics.³ Although the numerical bifurcation values are different, qualitatively this bifurcation scenario corresponds exactly to the bifurcation scenario under the Walrasian equilibrium setting as in BH (Lemma 3, p. 1249).

For $z_s = 0$ the system is symmetric with respect to the m-axis (x = 0) and the Neimark-Sacker bifurcation of the symmetric non-fundamental steady states E_r and E_l occur for the same parameter $\beta = \beta_{NS}$. For $z_s > 0$ the system is not symmetric anymore, and z_s is thus a symmetry breaking parameter of the system. We have the following result:

Proposition 3. (Stability & bifurcations of fundamental/non-fundamental steady states for $z_s > 0$; see figure 2) For R < g < 2R and $0 < \mu < \frac{2}{R}$, let $\beta_{sn} = \frac{8 \tanh^{-1}(\frac{2R}{g}-1)(R-1)}{4(R-1)C+gz_s^2}$, $x_r = \frac{z_s}{2(R-1)} + \sqrt{\frac{C - \frac{2}{\beta} \tanh^{-1}(\frac{2R}{g}-1)}{g(R-1)} + \frac{z_s^2}{4(R-1)^2}}, x_l = \frac{z_s}{2(R-1)} - \sqrt{\frac{C - \frac{2}{\beta} \tanh^{-1}(\frac{2R}{g}-1)}{g(R-1)} + \frac{z_s^2}{4(R-1)^2}}.$ We have:

- (i) $0 \leq \beta < \beta_{sn}$: the fundamental steady state $E = (0, -\tanh(\frac{\beta C}{2}))$ is globally stable;
- (ii) $\beta = \beta_{sn}$: a saddle-node bifurcation occurs and there is exactly one non-fundamental steady state;
- (iii) $\beta_{sn} < \beta < \beta^*$: E is locally stable and there are two non-fundamental steady states $E_r = (x_r, 1 \frac{2R}{g})$ and $E_l = (x_l, 1 \frac{2R}{g})$ with $0 < x_l < x_r$; (iv) $\beta = \beta^*$: a transcritical bifurcation occurs and the non-fundamental steady state E_l
- (iv) $\beta = \beta^*$: a transcritical bifurcation occurs and the non-fundamental steady state E_l coincides with the fundamental steady state E;
- (v) $\beta > \beta^*$: E is unstable and there are two non-fundamental steady states $E_r = (x_r, 1 \frac{2R}{g})$ and $E_l = (x_l, 1 \frac{2R}{g})$ with $x_l < 0 < x_r$.

Moreover, when z_s is positive but small, there exists $\beta_r > \beta_{sn}$ and $\beta_l > \beta^*$ such that

- (vi) E_r is locally stable for $\beta_{sn} < \beta < \beta_r$ and unstable for $\beta > \beta_r$; a supercritical Neimark-Sacker bifurcation occurs at $\beta = \beta_r$ for $\alpha < 0.6$;
- (vii) E_l is unstable for $\beta_{sn} < \beta < \beta^*$, locally stable for $\beta^* < \beta < \beta_l$ and unstable for $\beta > \beta_l$; a transcritical bifurcation occurs at $\beta = \beta^*$, for which $x_l = 0$, and a supercritical Neimark-Sacker bifurcation occurs at $\beta = \beta_l$ for $\alpha < 0.6$.



Figure 2: Bifurcation diagrams of (a) the symmetric case $z_s = 0$ and (b) the asymmetric case $z_s > 0$.



Figure 3: Supercritical Neimark-Sacker bifurcation for $R = 1.1, g = 1.15, \mu = 1.6, \alpha = 0.5, C = 1$, (a) the symmetric case: $\beta_{NS} \approx 4.18006$, (b) the asymmetric case: $\beta_r \approx 4.11408, \beta_l \approx 4.24672$.

Proposition 3 is illustrated in figure 2. For $z_s = 0$ the bifurcation diagram is symmetric with respect to x = 0. Symmetry breaking occurs for $z_s > 0$, and the non-generic pitchfork bifurcation is replaced by the generic co-dimension-1 saddle-node bifurcation and a transcritical bifurcation. Since for $z_s = 0$, E_r and E_l will undergo supercritical Neimark-Sacker bifurcation as β increases, we know that for small $z_s > 0$, E_r and E_l will also undergo supercritical Neimark-Sacker bifurcation as β increases because the occurence of the Neimark-Sacker bifurcation is a persistent property. Thus there also exist two attracting invariant circles after the supercritical Neimark-Sacker bifurcations in the asymmetric case. Moreover, from the explicit expressions for β_r and β_l (see Appendix) it follows that $\beta_r < \beta_{NS} < \beta_l$. So in the asymmetric case, the two Neimark-Sacker bifurcations of E_r and E_l do not occur simultaneously, as illustrated in figure 3.

4 Global Dynamics

This section discusses the global dynamical behaviour of the asset pricing model with fundamentalists and trend followers. We start with some numerical simulations in Subsection

³For the system with strong asynchronous updating of beliefs, i.e. large α , the Neimark-Sacker bifurcation becomes subcritical. We will restrict the analysis to $\alpha < 0.6$ and compare the results to the case of synchronous updating (i.e. $\alpha = 0$). The other cases are left for future work.

4.1. Subsection 4.2 discusses the stable and the unstable manifolds of the fundamental steady state, which play a key role in understanding the global dynamics. Complicated dynamics and strange attractors are discussed in Subsection 4.3.

4.1 Numerical Simulations

We investigate the global dynamical behaviour as the intensity of choice β becomes larger. In all simulations the other parameters are fixed at

$$R = 1.1, g = 1.15, \mu = 1.6, \alpha = 0.5, C = 1, a\sigma^2 = 1 \text{ and } z_s = 0.01.$$

We emphasize that for other choices of the parameter values the results are similar. We have chosen these particular values for various reasons, in particular for analytical tractibility and expositionary purposes. Let us briefly discuss each parameter separately:

- R = 1.1. We have chosen R = 1.1 in order to compare the market maker scenario directly to the Walrasian scenario studied in BH 1998, who also used R = 1.1. When the model is viewed on a daily time scale, the value of R is very close to 1. In any case R > 1, since otherwise the risky asset does not have a finite fundamental price $p^* = \bar{y}/(R-1)$. For R > 1, and very close to 1, the results are similar but the corresponding figures illustrating the route to complicated dynamics are less clear and harder to read. From a didactical viewpoint, R = 1.1 yields better graphs explaining and illustrating the typical bifurcation route to complicated dynamics;
- g = 1.15. This trend parameter g satisfies the inequality $R < g < R^2$ of proposition 4 below. For this range of g-values the same analysis applies. The lower bound g > R implies that trend followers believe that asset prices grow faster than at the rate of the risk free asset, ensuring that the fundamental steady state becomes unstable for large intensity of choice β . The upper bound $g < R^2$ ensures that the dynamics remains bounded. Without this upper bound, the system may lock into a self-fulfilling bubble solution with trend followers earning higher profits and the asset price diverging to infinity. One could extend the model, and allow for larger g-values, by introducing some 'stabilizing force' to prevent asset pricing from diverging to plus infinity (see e.g. De Grauwe et al. (1993)). Such an additional, stabilizing force would create an additional nonlinearity and make the analysis more complicated. We conjecture that such an extended model yields similar dynamical behaviour for $g > R^2$ also, but an analysis is beyond the scope of the current paper;
- $\mu = 1.6$. The parameter μ is the speed of adjustment at which the market maker adjusts asset prices, and thus reflects the market micro structure. The model represents a stylized price adjustment process, satisfying the law of demand and supply, but it is difficult to say exactly what a reasonable value of μ is. It should be emphasized however, that similar results hold for a range of μ -values. We focus on the case $1/R < \mu < 2/R$ which is analytically most tractible, and where a rational route to randomness will be explained by a homoclinic bifurcation between the stable and unstable manifolds of the fundamental steady state. Similar bifurcation routes to complicated dynamics have also been observed numerically for $0 \le \mu \le$

1/R. We conjecture that in that case a rational route to randomness occurs due to homoclinic bifurcations between the stable and unstable manifolds of *periodic* saddle points, but a detailed analysis is beyond the scope of this paper;

- $\alpha = 0.5$. This parameter reflects the speed of asynchronous updating. For $\alpha = 0.5$, 50% of all agents updates their strategy each period. Note that α is an eigenvalue of the Jacobian matrix of the steady state, characterizing the speed of convergence in periods where prices move towards the fundamental value. For $0 \le \alpha < 0.6$ the same analysis applies (see footnote 3 for the upperbound $\alpha = 0.6$).
- C = 1. Costs for fundamentalists have been normalized to 1.
- $a\sigma^2 = 1$. This is just a normalization, see footnote 2.
- $z_s = 0.01$. This parameter represents the outside supply of shares per trader. When the number of traders is large, z_s should be small. This parameter plays a role in the symmetry breaking of the system.



Figure 4: Bifurcation diagram and Lyapunov exponent for (a) an initial state near the negative non-fundamental steady state E_l , and (b) an initial state near the positive non-fundamental steady state E_r .

Figure 4 shows bifurcation diagrams and the corresponding Lyapunov Characteristic Exponent(LCE) for increasing β , $4.0 \leq \beta \leq 5.0$. In the left panel, the initial state is near E_l , with $x_0 < 0$, while in the right panel the initial state is near E_r , with $x_0 > 0$. Figure 5 shows some typical time series of the price deviation x and Figure 6 shows the corresponding attractors. The behaviour for $4.0 \leq \beta \leq 5.0$ may be summarized as follows (see figures 4, 5 and 6):

• for $4.0 \leq \beta < 4.11408$ both non-fundamental steady states are locally stable; for $\beta \approx 4.11408$, the positive steady state E_r undergoes a supercritical Neimark-Sacker bifurcation;



Figure 5: Time series for $\beta = 4.3$, 4.45, 4.6, for (a) negative (below the fundamental) initial state (left panel), and (b) positive (above the fundamental) initial state (right panel).

• for $4.11408 < \beta < 4.24672$ the locally stable negative non-fundamental steady state E_l coexists with a periodic or quasi-periodic attractor around the positive unstable non-fundamental steady state E_r ;

for $\beta \approx 4.24672$, the negative steady state E_r undergoes a supercritical Neimark-Sacker bifurcation;

- for $4.24672 < \beta < 4.39288$ both non-fundamental steady states are unstable. Two attractors coexists, one (quasi-)periodic around the unstable negative steady state E_l and one periodic or quasi-periodic attractor around the positive unstable steady state E_r ; the latter attractor becomes chaotic, with positive Lyapunov exponent, as β approaches 4.39288; for $\beta \approx 4.39288$ the positive (chaotic) attractor suddenly disappears;
- for $4.39288 < \beta < 4.55327$ there seems to be only one attractor, around the locally unstable negative steady state E_l ; this attractor is first periodic or quasi-periodic, but becomes chaotic, with positive Lyapunov exponent, as β approaches 4.55327; for $\beta \approx 4.55327$ the negative (chaotic) attractor suddenly disappears;
- for $4.55327 < \beta \leq 5.0$ there seems to be only one attractor, with two branches, one around the unstable negative steady state E_l and a second branch around the unstable positive steady state E_r ; for many β values this attractor is chaotic, with positive Lyapunov exponent.

An important feature of the model is the co-existence of different attractors. In particular, an attractor representing an optimistic state of the market, i.e. with price



Figure 6: Attractors for different β -values: (a) $\beta = 4.3$: two co-existing (quasi-)periodic attractors; (b) $\beta = 4.38$: (quasi-)periodic attractor (left) coexists with chaotic attractor (right); (c) $\beta = 4.39288$: chaotic attractor becomes 'tangent' to stable manifold (the vertical line x = 0) of the fundamental steady state; (d) $\beta = 4.4$ single (quasi-)periodic attractor; (e) $\beta = 4.55327$: single chaotic attractor becomes 'tangent' to stable manifold (the vertical line x = 0) of the fundamental steady state; (d) $\beta = 4.65$: chaotic attractor on both sides of fundamental steady state.

fluctuations above the fundamental value, can coexists with an attractor representing a pessimistic state, i.e. with prices fluctuating below their fundamental value. This feature also occurs in the BH-model with a Walrasian market equilibrium scenario. However, in contrast to the BH-framework, as the intensity of choice further increases our model with a market maker exhibits *endogenous switching between optimistic and pessimistic states*, as illustrated for example in the time series for $\beta = 4.6$ in figure 5. Under the Walrasion equilibrium scenario switching between optimistic and pessimistic states only occurs due to exogenous noise or after adding more trader types to the market. The market maker institution thus leads more easily to switching between optimistic and pessimistic market outcomes, even in a simple setting with fundamentalists versus trend followers.

4.2 The Stable and Unstable Manifolds

What causes the sudden disappearance of the attractors representing optimistic or pessimistic states of the market? It turns out that homoclinic bifurcations between the stable and unstable manifold of the fundamental steady state play a key role in explaining the endogenous switching between optimistic and pessimistic market states. For convenience of the reader, we briefly recall some basic notions.

Consider a differentiable two-dimensional map $\Psi_{\beta} : \mathbb{R}^2 \to \mathbb{R}^2$, where $\beta \in \mathbb{R}$ is a parameter. Let p be a saddle fixed with real eigenvalues λ_1 and λ_2 such that $0 < |\lambda_2| < 1 < |\lambda_1|$. Moreover p is called dissipative if $|\lambda_1\lambda_2| < 1$. The local stable set or stable manifold of the fixed point p is

$$W_{loc}^{s}(p) = \{ x \in U | \Psi^{n}(x) \to p \text{ as } n \to +\infty \},\$$

and the local unstable set or unstable manifold is

$$W_{loc}^{u}(p) = \{ x \in U | \Psi^{n}(x) \to p \text{ as } n \to -\infty \},\$$

where U is some small neighborhood of p. The global stable set or stable manifold and the global unstable set or unstable manifold are defined as

$$W^{s}(p) = \bigcup_{n \le 0} \Psi^{n}(W^{s}_{loc}(p)) \quad and \quad W^{u}(p) = \bigcup_{n \ge 0} \Psi^{n}(W^{u}_{loc}(p)).$$

A point $q \neq p$ is called a homoclinic point if $q \in W^u(p) \cap W^s(p)$; moreover q is called a transversal homoclinic point if $W^u(p)$ and $W^s(p)$ intersect transversely at q, while qis called a point of homoclinic tangency if $W^u(p)$ and $W^s(p)$ intersect tangentially at q. The notion of homoclinic orbit was introduced by Poincaré already more than a century ago, who realized that existence of a homoclinic point implies complicated dynamical behaviour.

In our model, the vertical *m*-axes, $(p = 0, -1 \le m \le 1)$, belongs to the stable manifold of the fundamental steady state $E = (0, -\tanh(\frac{\beta C}{2}))$. Since the horizontal line $m = m_0$, with $m_0 = 1 - \frac{2R}{g} + \frac{2}{\mu g}$, is mapped onto the m-axis, this horizontal line also lies in the stable manifold of E. Figure 7 shows the unstable manifold of the fundamental steady state for increasing values of the parameter β . The unstable manifold of the fundamental steady state has two branches, one in the positive and one in the negative half plane, spiralling around the non-fundamental steady states. Each region bounded by a branch

of the unstable manifold yields an invariant region and the two co-existing attractors are contained in these regions (see e.g. the upper panel in figure 7). For $\beta \approx 4.39288$ a first homoclinic bifurcation occurs between the positive branch of the unstable manifold and the stable manifold (see figure 7c and its enlargement in figure 7g). At this critical homoclinic bifurcation value, the positive region still contains an attractor. After this first homoclinic bifurcation however, points escape from the positive half plane into the negative half plane and converge to the negative attractor, as illustrated in figure 6d. As β further increases, for $\beta \approx 4.55327$, a second homoclinic tangency occurs, this time between the left branch of the unstable manifold and the stable manifold, as illustrated in figure 7e. At the homoclinic bifurcation value, the left region is still invariant and contains an attractor below the fundamental price. For $\beta > 4.55327$ both the left and the right branch have transversal intersections with the stable manifold, and points now can also escape from left to right. Both branches of the unstable manifold oscillate wildly, as illustrated in figures 7f and its enlargement Figure 7h, and the attractor is neither contained in the negative nor in the positive region, but rather lies in both regions. The bifurcation route to endogenous switching between optimistic and pessimistic market states is thus explained by homoclinic bifurcations of the stable and unstable manifolds of the fundamental steady state.

4.3 Strange Attractors

The existence of transversal homoclinic orbits implies complicated dynamical behaviour in the system, such as existence of the well known Smale horseshoe chaotic invariant sets. Smale (1965) proved that, for a diffeomorphism Ψ with a transverse homoclinic point, there exists a suitable integer k such that Ψ^k has invariant set Λ in the neigbourhood of the homoclinic point, with infinitely many periodic points and, an uncountable set of aperiodic points as well as orbits which are dense in Λ . Moreover, the map Ψ^k restricted to Λ exhibits sensitivity to initial states and therefore the dynamics is chaotic. See also Palis and Takens (1993, esp. Chapter 2).⁴

For the system we are considering, the fundamental steady state E becomes a saddle point after the pitchfork bifurcation in the symmetric case $z_s = 0$ or after the transcritical bifurcation in the asymmetric case $z_s > 0$. Figures 7c and 7g show that, as the intensity of choice β increases, a homoclinic bifurcation occurs, between the right respectively the left branch of the unstable manifold $W^u(E)$ of and the stable manifold $W^s(E)$ of the fundamental steady states. A number of complicated phenomena occur due to the homoclinic bifurcation, such as existence of strange attractor, co-existence of infinitely many stable cycles and cascades of infinitely many period doubling and period halving bifurcations; see Palis and Takens (1993) for an extensive mathematical treatment. Here we focus on existence of strange attractors. An important result is the following:

Strange Attractor Theorem (see Mora and Viana (1993); see also Palis and Takens (1993)) Let $\mathcal{F}_{\beta} : \mathbb{R}^2 \to \mathbb{R}^2$ be a two-dimensional map with parameter β , and let p be a dissipative saddle point. If the map \mathcal{F}_{β} exhibits a generic homoclinic bifurcation between

 $^{^{4}}$ Hale and Lin (1986) show that these properties are of a local nature and can be generalized to endomorphism (i.e. non-invertible maps) with continuous differential. Gardini (1996) presents a case study that the result may also hold for some endomorphism on the plane without continuous derivative.



Figure 7: Unstable manifold of E for $\beta = (a)$ 4.3, (b) 4.38, (c) 4.39288, (d) 4.4, (e) 4.55327, (f) 4.65, (g) 4.39288 in detail, (h) 4.65 in detail. Homoclinic bifurcations between the stable and unstable manifolds of the fundamental steady state occur for $\beta \approx 4.39288$ (Figure 7c and its enlargement in 7g) and for $\beta \approx 4.55327$ (Figure 7e).

the stable manifold and the unstable manifold of the saddle point at $\beta = \beta_h$, then there exists a positive Lebesgue measure set $\Omega \subset (\beta_h - \epsilon, \beta_h + \epsilon)$, such that for all $\beta \in \Omega$ the map \mathcal{F}_{β} has a strange attractor.

Roughly speaking the theorem states that close to a homoclinic bifurcation strange attractors arise with positive probability in the parameter space.

Figures 7c and 7g already suggest that this result also holds for our model, in the asymmetric case $z_s > 0$. We state and prove (see the appendix) the result here only for the symmetric case $z_s = 0$, since the computations are considerably simplified in the symmetric case; the main ideas underlying the proof suggest that similar results also hold in the asymmetric case.⁵

Proposition 4. (Homoclinic bifurcation of fundamental steady state for $z_s = 0$) Assume $R < g < R^2$, $\frac{1}{R} < \mu < \min\{\frac{2}{R}, \frac{R-1}{g-R}\}$, $0 < \alpha < \frac{\sqrt{\mu R-1}}{\sqrt{\mu g}}$. Then the fundamental steady state E is a dissipative saddle point for all $\beta > \beta^*$ after the pitchfork bifurcation. Moreover,

- (i) for $\beta > \beta^*$ with $|\beta \beta^*|$ small, there is no intersection between $W^u(E)$ and $W^s(E)$ (the unstable manifold of E is attracted by E_r and E_l);
- (ii) for some $\beta_h > \beta^*$, a homoclinic bifurcation between the stable and unstable manifolds of E occurs;
- (iii) for $\beta > \beta_h$, there always exist transversal homoclinic orbits of E.

Corollary 4.1 (Existence of strange attractors) For the other parameters fixed as in proposition 4, there exists a positive Lebesgue measure set of β -values in the parameter interval $(\beta_h - \epsilon, \beta_h + \epsilon)$ for which the dynamic system generated by the map \mathcal{F} has a strange attractor.

Corollary 4.2 (Existence of chaotic orbits) For the other parameters fixed as in proposition 4 and β sufficiently large, there exists an invariant Cantor set containing an uncountable set of initial states with chaotic orbits.

According to corollary 4.2, when the intensity of choice is sufficiently large, the model has an invariang Cantor set with many chaotic orbits. Typically these chaotic orbits have a saddle structure, and are thus unstable so that the long run dynamical behaviour typically may still be regular. According to corrollary 4.1 however, strange attracors occur in the model for a large set of β -values. One may interpret this result as saying that strange attractors occur with positive probability in the parameter space.

⁵We restrict our analysis to the symmetric case $z_s = 0$ and other parameter intervals as stated in the proposition, which include e.g. our simulation benchmark $R = 1.1, g = 1.15, \mu = 1.6, \alpha = 0.5$. Homoclinic bifurcations between the stable and unstable manifolds of the fundamental steady state occur for a much wider range of parameter values, but the proof would require considering many different detailed subcases without gaining much insight. We therefore focus on the cases stated in the proposition. Furthermore, it should be clear from Figure 7 that the horizontal line $m_0 = 1 - \frac{2R}{g} + \frac{2}{\mu g}$, which is part of the stable manifold of the fundamental steady state E, plays an important role. The reader may easily verify that $m_0 < 1$ if and only if $\mu > 1/R$. For $\mu > 1/R$ a homoclinic bifurcation between the stable and unstable manifolds of the fundamental steady state occur as the intensity of choice increases. Bifurcation routes to complicated dynamics have also been observed numerically for $0 \le \mu \le 1/R$. We conjecture that in that case a rational route to randomness occurs due to homoclinic bifurcations between the stable and unstable manifolds of *periodic* saddle points, but a detailed analysis is much more complicated and beyond the scope of this paper.

5 Conclusion

We have investigated an evolutionary asset pricing model with speculative traders, fundamentalists and technical analysts (trend followers), where agents tend to choose strategies that have performed well, according to realized profits, in the recent past. Our model deviates in three important ways from the adaptive belief systems of Brock and Hommes (1998). Firstly, intead of the Walrasian equilibrium price scenario, we considered a model with a market maker and a simple price adjustment rule, where price changes are proportional to excess demand. Secondly, we allow for asynchronous updating of strategies, instead of synchronous updating of beliefs. In our framework, in each period only a (fixed) fraction of the traders updates their beliefs. A third extension of our model has been that we have been considering the case of positive outside supply of shares. In that case, the fundamental price is affected by a risk premium, which also shows up in the fitness measure, and therefore affects evolutionary dynamics.

Summarizing the results, one may say that despite three important changes in the model, the dynamical behaviour is surprisingly similar to the BH dynamics. In particular, in a world where fundamentalist strategies are more costly than trend following strategies, a rational route to randomness, that is, a bifurcation route to strange attractors, occurs as the sensitivity to switch strategies increases. Although some details of this route may be different, the global picture remains the same, with creation of two non-fundamental steady states, Neimark-Sacker bifurcations of these non-fundamental steady states, invariant circles around each of the non-fundamental steady states and breaking of these invariant circles into a strange attractor. There is one difference in the dynamical behaviour worthwhile mentioning. In contrast to the Walrasian market equilibrium scenario, under a market maker scenario, in a world with costly fundamentalism and free technical trading endogenous switching between "optimistic states" where the market is overvalued and "pessimistic states" where the market is undervalued can occur when the intensity of choice increases. The main conclusion from our analysis however is that global dynamical features do not necessarily depend upon details of the market institution model, and can be quite robust with respect to changes in the specifications of the functional equations and modelling assumptions.

Appendix

Proof of Proposition 1. The Jacobian matrix at fundamental steady state E is

$$J(\beta) = \begin{pmatrix} v(-\tanh(\frac{\beta C}{2})) & 0\\ 0 & \alpha \end{pmatrix}$$

with eigenvalues $\lambda_1 = v(-\tanh(\frac{\beta C}{2}))$ and $\lambda_2 = \alpha$. Since $0 \le \alpha < 1$, the stability of E dependends on λ_1 . Notice that $-1 < \lambda_1 < 1$ is equivalent to $\frac{2R}{g} - 1 - \frac{4}{\mu g} < \tanh(\frac{\beta C}{2}) < \frac{2R}{g} - 1$. Local stability of E may then be verified by direct calculation. Moreover, it is clear that E is globally stable for 0 < g < R and $0 < \mu < \frac{2}{R}$. The global stability of E for R < g < 2R and $0 < \mu < \frac{2}{R}$ can be verified by noticing that points below the line $m = 1 - \frac{2R}{g}$ will be mapped above this line in finite iterations and then will be attracted by E.

Let C_m be the locus of the points satisfying $F_2(x,m) = m$, that is

 $C_m = \{(x, m) \mid m = \tanh(\beta w(x, m))\}.$

The birth of a two-cycle at infinity for $\mu = \frac{2}{R}$ and the period-doubling bifurcation at $\beta = \beta_{pd}$ for $\mu > \frac{4}{2R-g}$ can be verified by noticing that the points of the 2-cycle correspond to the two intersection points of the line $m = 1 - \frac{2R}{g} + \frac{4}{\mu g}$ and the curve C_m . By direct calculation it can be shown that the 2-cycle created in the period-doubling bifurcation is unstable.

Now consider the existence of non-fundamental steady state for R < g < 2R and $0 < \mu < \frac{4}{2R-g}$. It is clear that (x,m) with $x \neq 0$ is a non-fundamental steady state if and only if (x,m) is an intersection point of the line $m = 1 - \frac{2R}{g}$ and C_m . Thus the result of the pitchfork bifurcation at $\beta = \beta^*$ can be readily verified.

Proof of Proposition 2. Since the system is symmetric for $z_s = 0$, we just focus on the positive non-fundamental steady state E_r . Notice that, with the other parameters fixed, as β increases from β^* to ∞ , E_r moves from $(0, 1 - \frac{2R}{g})$ to $(\sqrt{\frac{C}{g(R-1)}}, 1 - \frac{2R}{g})$ along the line $m = 1 - \frac{2R}{g}$. The Jacobian matrix at E_r is

$$J_{r}(\beta) = \begin{pmatrix} 1 & -\frac{\mu g}{2}x_{r} \\ (1-\alpha)4\beta g^{-1}R(R-1)(g-R)x_{r} & \alpha + (1-\alpha)\beta\mu R(g-R)x_{r}^{2} \end{pmatrix}.$$

The trace and the determinant of J_r are

$$\mathrm{tr}J_r = 1 + \alpha + (1 - \alpha)\beta\mu R(g - R)x_r^2,$$

$$\det J_r = \alpha + (1-\alpha)\beta\mu R(g-R)(2R-1)x_r^2.$$

The characteristic equation is

 $Q(\lambda) = \lambda^2 - \mathrm{tr} J_r \lambda + \mathrm{det} J_r = 0.$

Let λ_1 and λ_2 be the eigenvalues. For $\beta > \beta^*$ it follows from $x_r > 0$ that $\lambda_1 \lambda_2 = \det J_r > 0$, so λ_1 and λ_2 are on the same side of the complex plane. Since $\lambda_1 + \lambda_2 = \operatorname{tr} J_r > 1 + \alpha$, it is clear that λ_1 and λ_2 are both on the right side of the complex plane. For $\beta = \beta^*$, $\lambda_1 = 1$ and $\lambda_2 = \alpha < 1$. Notice that Q(1) > 0 and $\frac{dQ(1)}{d\beta} > 0$ for all $\beta > \beta^*$. Therefore $0 < \lambda_2 < \lambda_1 < 1$ when $\beta > \beta^*$ with $|\beta - \beta^*|$ small and 1 is never an eigenvalue for $\beta > \beta^*$. Since $\frac{d(\lambda_1 \lambda_2)}{d\beta} = \frac{d(\det J_r)}{d\beta} > 0$ for $\beta > \beta^*$, and $\det J_r \to \infty$ as $\beta \to \infty$, there exits a $\beta_{NS} > \beta^*$ such that E_r is locally stable for $\beta^* < \beta < \beta_{NS}$ and E_r is unstable for $\beta > \beta_{NS}$.

Let $x_0 = x_r$, $J_0 = J_r$ at $\beta = \beta_{NS}$. It can be verified that

$$\beta_{NS} = \frac{2}{C} \tanh^{-1}(\frac{2R}{g} - 1) + \frac{g(R - 1)}{\mu R(g - R)(2R - 1)C},$$
$$x_0 = \frac{\sqrt{C}}{\sqrt{g(R - 1) + 2\tanh^{-1}(\frac{2R}{g} - 1)\mu R(g - R)(2R - 1)}}.$$

Moreover $\det J_0 = 1$ and the characteristic equation is

$$\lambda^2 - \mathrm{tr}J_0\lambda + 1 = 0$$

with $tr J_0 = \frac{2R+2(R-1)\alpha}{2R-1}$. So the eigenvalues are

$$\lambda_{1,2} = e^{\pm i\theta} = \frac{1}{2} \operatorname{tr} J_0 \pm i \sqrt{1 - (\frac{1}{2} \operatorname{tr} J_0)^2},$$

with $|\lambda_1| = |\lambda_2| = 1$. It is clear that $0 < \theta < \frac{\pi}{3}$ and therefore $\lambda_{1,2}$ satisfies the nonresonance condition $\lambda_{1,2} \notin \{e^{2\pi i \frac{p}{q}} : p, q = 1, \cdots, 6\}$. Moreover from the above discussion it is clear that $\frac{d|\lambda_1|}{d\beta} = \frac{d|\lambda_2|}{d\beta} > 0$ at $\beta = \beta_{NS}$. To verify that the Neimark-Sacker bifurcation is supercritical its normal form has to

To verify that the Neimark-Sacker bifurcation is supercritical its normal form has to be computed; see Kuznetsov (1998), especially section 4.7, for an extensive treatment of bifurcation theory including the Neimark-Sacker bifurcation. First choose the eigenvector q, which satisfies $J_0q = e^{i\theta}q$, as

$$q = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\mu g x_0 \\ 1 - e^{i\theta} \end{pmatrix},$$

while the adjoint eigenvector p, which satisfies $J_0{}^T p = e^{-i\theta}p$, may be chosen as

$$p = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} \frac{(g-R)R(2R-1)^2}{g(3R-1+(R-1)\alpha)}\beta_{NS}x_0(1+e^{i\theta}) \\ \frac{(2R-1)^2}{4(1-\alpha)(R-1)(3R-1+(R-1)\alpha)}(e^{-i\theta}-e^{i\theta}) \end{pmatrix}.$$

It can be seen that $\langle q, p \rangle = \bar{q_1}p_1 + \bar{q_2}p_2 = 1$. Then we compose

$$\begin{pmatrix} x\\m \end{pmatrix} = \begin{pmatrix} x_0 + q_1 z + \bar{q_1}\bar{z}\\1 - \frac{2R}{g} + q_2 z + \bar{q_2}\bar{z} \end{cases}$$

and evaluate the function

$$H(z,\bar{z}) = \bar{p}_1(F_1(x_0 + q_1z + \bar{q}_1\bar{z}, 1 - \frac{2R}{g} + q_2z + \bar{q}_2\bar{z}) - x_0) + \bar{p}_2(F_2(x_0 + q_1z + \bar{q}_1\bar{z}, 1 - \frac{2R}{g} + q_2z + \bar{q}_2\bar{z}) - (1 - \frac{2R}{g}))$$

at $\beta = \beta_{NS}$. Computing its Taylor expansion at $(z, \bar{z}) = (0, 0)$ gives

$$H(z, \bar{z}) = e^{i heta} z + \sum_{2 \le j+k \le 3} rac{1}{j!k!} g_{jk} z^j \bar{z}^k + O(|z|^4).$$

A straightforward computation then shows that the critical real part

$$a(\beta_{NS}) = Re(\frac{e^{-i\theta}g_{21}}{2}) - Re(\frac{(1-2e^{i\theta})e^{-2i\theta}}{2(1-e^{i\theta})}g_{20}g_{11}) - \frac{1}{2}|g_{11}|^2 - \frac{1}{4}|g_{02}|^2 < 0$$

for 1 < R < 1.2, R < g < 2R and $0 < \alpha < 0.6$. This implies that a supercritical Neimark-Sacker bifurcation occurs and a unique and stable closed invariant curve bifurcates from E_r for $\beta > \beta_{NS}$.

Proof of Proposition 3. For $z_s > 0$ the Jacobian matrix at the fundamental steady state E is

$$J(\beta) = \begin{pmatrix} v(-\tanh(\frac{\beta C}{2})) & 0\\ -\frac{(1-\alpha)\beta g z_s}{2}(\operatorname{sech}(\frac{\beta C}{2}))^2 & \alpha \end{pmatrix}$$

with eigenvalues $\lambda_1 = v(-\tanh(\frac{\beta C}{2}))$ and $\lambda_2 = \alpha$. The local stability of E can be verified by direct calculation. The global stability of E for $0 \leq \beta < \beta_{sn}$ can be verified by noticing that the points below the line $m = 1 - \frac{2R}{g}$ will be mapped above this line in finite iterations and then will be attracted by E.

Notice that (x, m) with $x \neq 0$ is a non-fundamental steady state if and only if (x, m) is a intersection point of the line $m = 1 - \frac{2R}{g}$ and C_m . Thus the result of the saddle-node bifurcation at $\beta = \beta_{sn}$ can be readily verified. If β increases from β_{sn} to ∞ , E_r moves from $(\frac{z_s}{2(R-1)}, 1 - \frac{2R}{g})$ to $(\frac{z_s}{2(R-1)} + \sqrt{\frac{C}{g(R-1)} + (\frac{z_s}{2(R-1)})^2}, 1 - \frac{2R}{g})$ along the line $m = 1 - \frac{2R}{g}$. The Jacobian matrix at E_r is

$$J_r(\beta) = \left(\begin{array}{cc} 1 & -\frac{\mu g}{2} x_r \\ (1-\alpha) \frac{4\beta}{g} R(R-1)(g-R)(x_r - \frac{z_s}{2(R-1)}) & \alpha + (1-\alpha)\beta\mu R(g-R)x_r^2 \end{array}\right)$$

with

$$\mathrm{tr}J_r = 1 + \alpha + (1 - \alpha)\beta\mu R(g - R)x_r^2,$$

$$\det J_r = \alpha + (1-\alpha)\beta\mu R(g-R)((2R-1)x_r^2 - z_s x_r).$$

Let λ_1 and λ_2 be the eigenvalues of the characteristic equation

$$Q(\lambda) = \lambda^2 - \mathrm{tr}J_r\lambda + \mathrm{det}J_r = 0$$

 $\begin{array}{l} \text{¿From } x_r > \frac{z_s}{2(R-1)} \text{ it follows that } \lambda_1 \lambda_2 = \det J_r > 0, \text{ so } \lambda_1 \text{ and } \lambda_2 \text{ are on the same side of the complex plane. Moreover, since } \lambda_1 + \lambda_2 = \operatorname{tr} J_r > 1 + \alpha, \text{ it is clear that } \lambda_1 \text{ and } \lambda_2 \text{ are both on the right side of complex plane. For } \beta = \beta_{sn}, \text{ it can be verified by direct computation that one of two eigenvalues of } E_r, \text{ say } \lambda_1 = 1; \text{ the other is } \lambda_2 = \det J_r. \\ \text{For } z_s \text{ small, such that } \frac{\tanh^{-1}(\frac{2R}{g}-1)\mu R(g-R)z_s^2}{(R-1)(4C(R-1)+gz_s^2)} < 1, \text{ it can be shown that } \lambda_2 < 1 \text{ for } \beta = \beta_{sn}. \\ \text{Notice that } Q(1) > 0 \text{ and } \frac{dQ(1)}{d\beta} > 0 \text{ for } \beta > \beta_{sn}. \\ \text{Therefore } 0 < \lambda_2 < \lambda_1 < 1 \text{ when } \beta > \beta_{sn}, \text{ with } |\beta - \beta_{sn}| \text{ small, and } 1 \text{ is never an eigenvalue for } \beta > \beta_{sn}. \\ \frac{d(\lambda_1\lambda_2)}{d\beta} = \frac{d(\det J_r)}{d\beta} > 0 \text{ for } \beta > \beta_{sn}, \text{ and } \det J_r \to \infty \text{ as } \beta \to \infty, \text{ there exists a } \beta_r > \beta_{sn} \text{ such that } E_r \text{ is locally stable for } \beta_{sn} < \beta < \beta_r \text{ and } E_r \text{ is unstable for } \beta > \beta_r. \end{array}$

Existence and stability of the negative steady state E_l can be discussed similarly. When β increases from β_{sn} to ∞ , E_l moves along the line $m = 1 - \frac{2R}{g}$ from $(\frac{z_s}{2(R-1)}, 1 - \frac{2R}{g})$ to $(\frac{z_s}{2(R-1)} - \sqrt{\frac{C}{g(R-1)} + (\frac{z_s}{2(R-1)})^2}, 1 - \frac{2R}{g})$. For $\beta = \beta^*$, E_l coincides with the fundamental steady state E at $(0, 1 - \frac{2R}{g})$. Moreover it can be verified that there exists a $\beta_l > \beta^*$ such that E_l is unstable for $\beta_{sn} < \beta < \beta^*$, locally stable for $\beta^* < \beta < \beta_l$, and unstable for $\beta > \beta_l$. We notice that the explicit expressions for β_r and β_l are

$$\beta_r = \frac{2 \tanh^{-1}(\frac{2R}{g} - 1)\mu R(g - R)(2R - 1) + g(R - 1)}{\mu R(g - R)((2R - 1)C + gRz_s x_r)},$$

$$\beta_l = \frac{2 \tanh^{-1}(\frac{2R}{g} - 1)\mu R(g - R)(2R - 1) + g(R - 1)}{\mu R(g - R)((2R - 1)C + gRz_s x_l)},$$

with $-x_0 < x_l < 0 < x_0 < x_r$, where x_0 is the x-value of the non-fundamental steady state corresponding to the Neimark-Sacker bifurcation in the symmetric case $z_s = 0$. Thus $\beta_r < \beta_{NS} < \beta_l$. Since the normal form computations of the Neimark-Sacker bifurcation depend continuously on the parameter z_s it follows that E_r and E_l also undergo supercritical Neimark-Sacker bifurcation at $\beta = \beta_r$ and $\beta = \beta_l$ respectively. In particular, for R = 1.1, g = 1.15, $\mu = 1.6$, $\alpha = 0.5$, C = 1, a straightforward computations shows that for $z_s = 0$ $a(\beta_{NS}) \approx -2.1591$ for both E_r and E_l , while for $z_s = 0.01 \ a(\beta_r) \approx -2.15237$ for E_r and $a(\beta_l) \approx -2.40127$ for E_l .

Proof of Proposition 4. It can be readily verified that the fundamental steady state E becomes a dissipative saddle fixed point after the pitchfork bifurcation. In fact, the two real eigenvalues of E are $\lambda_1(\beta) = v(-\tanh(\frac{\beta C}{2}))$ and $\lambda_2(\beta) = \alpha$. A simple computation then shows that E will be a dissipative saddle fixed point for all $\beta > \beta^*$ if $\alpha < \frac{1}{1+\mu(g-R)}$. This inequality follows from the assumptions on R, g, μ and α , since

$$\alpha < \frac{\sqrt{\mu R - 1}}{\sqrt{\mu g}} = \frac{\sqrt{R - \frac{1}{\mu}}}{\sqrt{g}} < \frac{\sqrt{\frac{R}{2}}}{\sqrt{g}} < \frac{1}{\sqrt{2}} < \frac{1}{R} < \frac{1}{1 + \mu(g - R)}.$$

For $R = 1.1, g = 1.15, \mu = 1.6$, the condition on α becomes $0 < \alpha < 0.64$.

Notice that immediately after the pitchfork bifurcation, the unstable manifold of E is attracted by E_r and E_l . Since the second eigenvalue of E is $0 < \lambda_2 = \alpha < 1$ at $\beta = \beta^*$, the dynamical system near the pitchfork bifurcation can be reduced to the one dimensional center manifold. At $\beta = \beta^*$ the two branches of $W^u(E)$ coincide with part of the center manifold and connect E with E_r and E with E_l . Thus for $\beta > \beta^*$ with $|\beta - \beta^*|$ small, there is no intersection between $W^u(E)$ and $W^s(E)$.

Next we show the existence of transversal homoclinic orbits for sufficiently large β . In order to understand the properties of the unstable manifold of E for large β , it will be useful to consider the limiting case $\beta = \infty$ first; see Figure 8 for illustration. In the limiting case the map (25) \mathcal{F}^{∞} becomes

$$F_1^{\infty}(x,m) = v(m)x,$$

$$F_2^{\infty}(x,m) = \begin{cases} \alpha m + (1-\alpha), & \text{for } w(x,m) > 0\\ \alpha m, & \text{for } w(x,m) = 0\\ \alpha m - (1-\alpha), & \text{for } w(x,m) < 0 \end{cases}$$

Let C_r and C_l be the locus of points satisfying w(x, m) = 0 on the right and left part of the plane respectively. Then C_r and C_l are

$$C_r = \{(x,m) \mid x(m) = \frac{\sqrt{C}}{\sqrt{g(R-v(m))}} > 0\},$$

$$C_l = \{(x,m) \mid x(m) = -\frac{\sqrt{C}}{\sqrt{g(R-v(m))}} < 0\}.$$

Notice that for $\beta = \infty$, the fundamental steady state is E = (0, -1). For C_r , x(m) is a decreasing function of m and it follows from $\mu < \frac{R-1}{g-R}$ that C_r and the line m = -1 intersect at the point $P_0 = (\frac{\sqrt{C}}{\sqrt{g(R-v(-1))}}, -1)$. The segment EP_0 lies in the unstable manifold of E. Let $P_{-1} = (\frac{\sqrt{C}(v(-1))^{-1}}{\sqrt{g(R-v(-1))}}, -1)$. Then $\mathcal{F}^{\infty}(P_{-1}) = P_0$. The first iterate of segment $P_{-1}P_0$ (excluding the end points) is a segment P_0P_1 with $P_1 = (\frac{\sqrt{C}v(-1)}{\sqrt{g(R-v(-1))}}, -1)$. The next iterate gives a segment P'_2P_2 with $P'_2 = (\frac{\sqrt{C}v(-1)}{\sqrt{g(R-v(-1))}}, 1 - 2\alpha)$ and $P_2 = (\frac{\sqrt{C}v(-1)}{\sqrt{g(R-v(-1))}}, 1 - 2\alpha)$.

 $(\frac{\sqrt{C}(v(-1))^2}{\sqrt{g(R-v(-1))}}, 1-2\alpha)$. It is clear that for $0 < \alpha < \frac{\mu R-1}{\mu g}$ the segment P'_2P_2 lies above the line $m = m_0$ (see Figure 8a, case 1), while for $\frac{\mu R-1}{\mu g} < \alpha < \frac{\sqrt{\mu R-1}}{\sqrt{\mu g}}$, P'_2P_2 lies between the line $m = m_0$ and $m = m_1$ with $m_1 = 1 - \frac{2R}{g}$ (see Figure 8b, case 2). Recall that the line $m = m_0$ belongs to the stable manifold of the fundamental steady state. In case 2 we consider the third iterate and it gives a segment P'_3P_3 with $P'_3 = (\frac{\sqrt{C}v(-1)v(1-2\alpha)}{\sqrt{g(R-v(-1))}}, 1-2\alpha^2)$

and $P_3 = (\frac{\sqrt{C}(v(-1))^2 v(1-2\alpha)}{\sqrt{g(R-v(-1))}}, 1-2\alpha^2)$, using the fact that P'_2P_2 is outside the region bounded by C_r and C_l . Clearly the segment P'_3P_3 lies above the line $m = m_0$. To understand the global geometric shape of the unstable manifold of E for large, but finite β , it is useful to investigate the jumping segment $P_1P'_2$ and its forward iterates. For β large, but finite, the unstable manifold of E will be close to the unstable manifold in the limiting case $\beta = \infty$. In the following we focus case 2 (Figures 8b-f); case 1 can be discussed in a similar way. Notice that the iterate of the segment $P_1P'_2$ under the map \mathcal{F}^{∞} just gives the segment $P_2P'_3$. Simple computation shows that the segment $P_2P'_3$ intersects with the line $m = m_0$ (i.e. the stable manifold of the fundamental steady state) at the point $Q^{\infty} = (\frac{\sqrt{C}v(-1)(\mu R-1)}{\sqrt{g(R-v(-1))}}(\frac{1}{\alpha}-1), m_0)$. For finite but large β , the unstable manifold of E is close to the polyline $P_oP_1P'_2P_2P'_3P_3$ and hence also transversaly intersects with the branch of the stable manifold $m = m_0$ at some point Q near the point Q^{∞} . Thus we have shown the existence of a transveral homoclinic point between the stable and the unstable manifolds of E for β sufficiently large.

By continuity it then follows that there exists a critical value $\beta_h > \beta^*$ such that a homoclinic bifurcation occurs at $\beta = \beta_h$.

Proof of Corollary 4.1. In order to apply the strange attractor theorem, certain generic conditions must be satisfyied. Takens(1992) presents three conditions for a generic system of homoclinic bifurcation: (i) \mathcal{F} is real analytic; (ii) the function $h(\beta) = -\frac{\ln |\lambda_1(\beta)|}{\ln |\lambda_2(\beta)|}$ is not constant, and (iii) the homoclinic tangency is inevitable, that is the system moves from a situation without any homoclinic points to the situation with transversal homoclinic points. All three generic conditions are satisfied in our system.

Finally, we notice that the strange attractor theorem applies to (local) diffeomorphisms. It is sufficient to show that in the region where the homoclinic bifurcation occurs the map \mathcal{F} is (locally) a diffeomorphism. Let $S = \{(x,m) | -1 < m < m_0\}$, where $m = m_0$ is part of the stable manifold of the fundamental steady state, and $S' = \mathcal{F}(S)$. We claim that $\mathcal{F} : S \to S'$ is a diffeomorphism. In fact it can be verified by direct calculation that for any point $p \in S$ the determinant of the Jacobian of the map \mathcal{F} is positive. Moreover, we claim that for any two points $p, q \in S$ with $p \neq q$ we have $\mathcal{F}(p) \neq \mathcal{F}(q)$. In fact, assume otherwise that $p = (x_1, m_1) \neq q = (x_2, m_2)$ such that $\mathcal{F}(p) = \mathcal{F}(q)$. From $F_1(x_1, m_1) = F_1(x_2, m_2)$ it follows that $m_1 \neq m_2$ since $p \neq q$. But then it also can be checked that $m_1 < m_2$ implies $F_2(x_1, m_1) < F_2(x_2, m_2)$ under the condition $\mu < \frac{R-1}{g-R}$. So this is a contradiction. Now consider the region above the line $m = m_0$.

Let J^0 be the locus of points where the Jacobian determinant of map \mathcal{F} vanish. Then it can be verified that J^0 is



Figure 8: Unstable manifold of E for $\beta = \infty$ vs. $\beta = 200$ with $R = 1.1, g = 1.15, \mu = 1.6, C = 1, z_s = 0.$

$$\mathbf{J}^{0} = \{(x,m) | \alpha v(m) + (1-\alpha) \frac{\mu g^{2} \beta}{4} (\operatorname{sech}(\beta w(x,m)))^{2} (2R - v(m))x^{2} = 0\}.$$

It can be verified that J^0 has three branches between $m_0 \le m \le 1$, which are symmetric with respect to the m-axes and are located strictly above the line $m = m_0$ except the point $(0, m_0)$ on the line, as illustrated in Figure 9. It then follows that in a neighbourhood where the homoclinic bifurcation occurs the map \mathcal{F} is locally a diffeomorphism.



Figure 9: Illustration of the curves J^0 , the stable and unstable manifold of E, near the homoclinic bifurcation, for (a) $\beta = 4.4724$, and (b) $\beta = 4.477$, with the ohter parameters fixed at R = 1.1, g = 1.15, $\mu = 1.6$, $\alpha = 0.5$, C = 1 and $z_s = 0$.

Proof of Corollary 4.2. We need to verify that for β large, the map \mathcal{F} is a local diffeomorphism close to the homoclinic orbit. We know from the proof of proposition 4 that, for sufficiently large β there is a transversal homoclinic point Q. \mathcal{F} maps the point Q onto a point on the m-axes strictly below the line $m = \alpha m_0 + (1 - \alpha)$, and the second iterate of Q will be on the m-axes strictly below the line $m = m_0$ for $0 < \alpha < \frac{\sqrt{\mu R-1}}{\sqrt{\mu g}}$. Notice that the only point of J_0 (i.e. the set of points where the determinant of the Jacobian matrix of \mathcal{F} vanishes) is $(0, m_0)$. Moreover when β tends to ∞ J⁰ moves close to the line $m = m_0$ and the part of the curves C_r and C_l between the line $m = m_0$ and m = 1. It can be verified that except for at most one value of α (for which $\mathcal{F}^{\infty}(Q^{\infty}) = (0, m_0)$) \mathcal{F} is locally invertible in the neighborhood of every point in the homoclinic orbit passing through Q, and therefore the unstable manifold intersect the stable manifold transversally at each point of the homoclinic orbit. Hence, we can apply the result of Hale and Lin (1986), implying the existence of chaotic orbits in the system for large β due to the existence of the transverse homoclinic orbits.

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