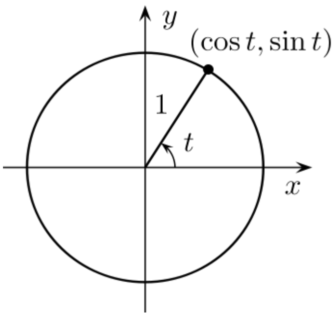
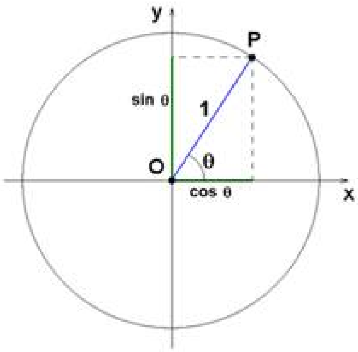
Trigonometric functions.

An important topic in all fields of science is the description of oscillations. Oscillations can be described very accurately by trigonometric functions. In physical science the phenomena as light, sound and electromagnetic waves use trigonometric functions. Also macro-economic theory deals with oscillatory movements in income, output and other variables. We now introduce three elementary trigonometric functions (sometimes also called "circular" functions), namely sine (comes from Latin: sinus, means "wave"), cosine and tangent, and we use as the function symbols for those functions the abbreviations: sin, cos and tan.

The introduction of the functions "sin" and "cos" can best be done by the unit circle, the circle with radius one as shown in the next figures.





In the XY-plane we look at a circle with midpoint (0,0) and radius 1. A point P(x,y) on the circle defines a line OP that makes an angle of t (or labeled with the Greek letter θ = theta in the second figure) degrees with the "base-line", that is the line that connects the points (0,0) with (1,0). As P moves along the unit circle in the positive (anti-clockwise) direction, the angle takes all values from 0° to 360°. As a first step we define the functions sin and cos as functions of the angle in degrees by:

$$\sin(t) = y$$
$$\cos(t) = x$$

Because P is a point on the unit circle, the distance of P(x,y) to (0,0) is always 1, so $y^2 + x^2 = 1$, and that gives us the obvious formula:

$$[\sin(t)]^2 + [\cos(t)]^2 = 1$$

Like most textbooks we will use the abbreviations: $\sin t$ for $\sin(t)$ and $\sin^2 t$ for $[\sin(t)]^2$ so that the previous formula will be denoted by:

$$\sin^2 t + \cos^2 t = 1$$

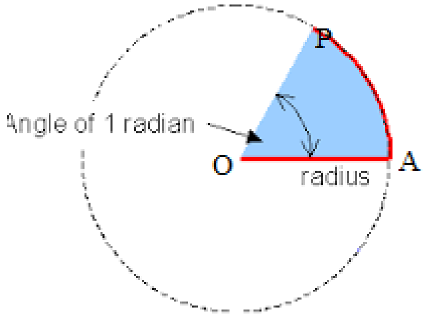
The next useful trigonometric function is the tangent, defined as the ratio of the sine to the cosine:

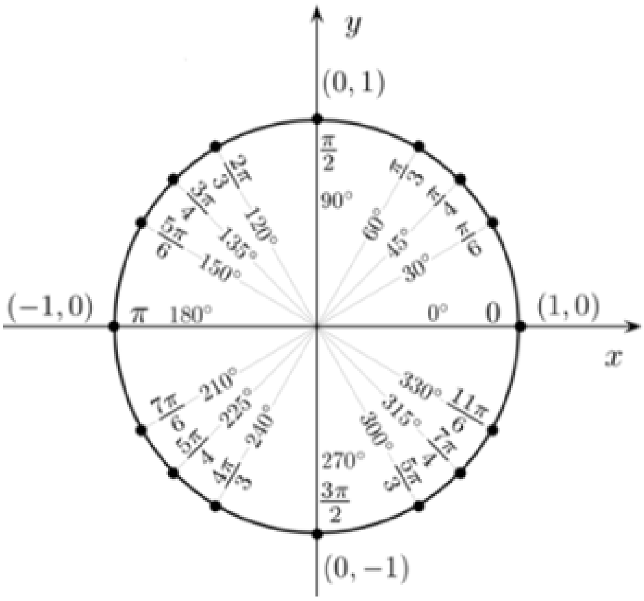
$$\tan t = \frac{\sin t}{\cos t}$$

This function is undefined for values of t that make cos t = 0. The graph of tan t has a vertical asymptote for t = 90° and t = 270°.
The next table gives us some particular values of the functions (as you can easily verify):

angle t	0°	90°	180°	270°	360°
sin t	0	1	0	– 1	0
cos t	1	0	– 1	0	1
tan t	0	± ∞	0	± ∞	0

For several reasons (especially for the use of those functions in calculus) we make a change of measurement in angles from degrees to units called radians. The new measure (in radians) of the angle AOP is the length of the arc AP taken counterclockwise round the circumference of the unit circle. The next figures will clarify:





Because the circumference of the total circle is 2π, that value corresponds to 360°. It is customary to use the variable x to denote angles in radians, although it should be emphasized that this x is a different x than used in the figures !

Hence to transform from degrees to radians or conversely: 2π rad $= 360°$.

so: x radians $= \dfrac{180}{\pi} x°$ and $\theta° = \dfrac{\pi}{180}\theta$ radians

It is useful to remember some particular angles:

$30° = \tfrac{1}{6}\pi$ rad

$45° = \tfrac{1}{4}\pi$ rad

$60° = \tfrac{1}{3}\pi$ rad

$90° = \tfrac{1}{2}\pi$ rad

$180° = \pi$ rad

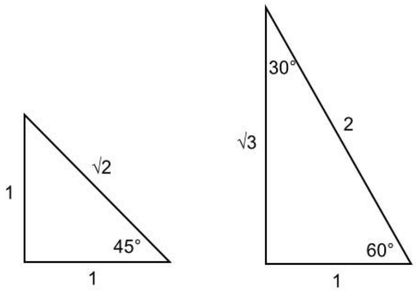
$360° = 2\pi$ rad

In all theoretical work, the radian measure of angles is used and denoted by a variable (such as x) without specific reference to the unit. The trigonometric functions are then sin x, cos x and

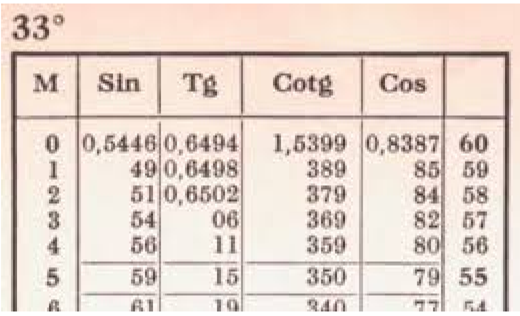
$$\tan x = \frac{\sin x}{\cos x}.$$

Trigonometric tables have been written already in early civilizations. In Europe the first accurate tables appeared in the 16th. century in Germany. For some special values of the angle x we can express the values in rationals or irrationals with the help of the square root and the well known triangles with 90/45/45 degrees and 90/30/60 degrees, see next figures. For instance:

$\sin 60° = \dfrac{\sqrt{3}}{2} = \tfrac{1}{2}\sqrt{3}$ but we prefer to write: $\sin \tfrac{1}{3}\pi = \dfrac{\sqrt{3}}{2} = \tfrac{1}{2}\sqrt{3}$



angle x	0	$\frac{1}{6}\pi$	$\frac{1}{4}\pi$	$\frac{1}{3}\pi$	$\frac{1}{2}\pi$	π	$\frac{3}{2}\pi$	2π
$\sin x$	0	$\frac{1}{2}$	$\frac{1}{2}\sqrt{2}$	$\frac{1}{2}\sqrt{3}$	1	0	-1	0
$\cos x$	1	$\frac{1}{2}\sqrt{3}$	$\frac{1}{2}\sqrt{2}$	$\frac{1}{2}$	0	-1	0	1
$\tan x$	0	$\frac{1}{3}\sqrt{3}$	1	$\sqrt{3}$	$\pm\infty$	0	$\pm\infty$	0


33°

M	Sin	Tg	Cotg	Cos	
0	0,5446	0,6494	1,5399	0,8387	60
1	49	0,6498	389	85	59
2	51	0,6502	379	84	58
3	54	06	369	82	57
4	56	11	359	80	56
5	59	15	350	79	55
6	61	19	340	77	54

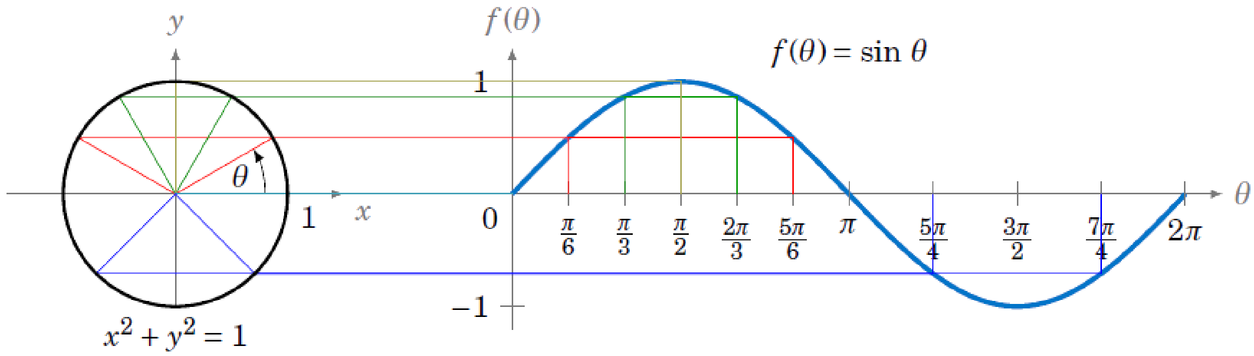
Here above you see a page of a trigonometric table book. It shows the values of Sin, Tg (= tan), Cotg (= 1/tan) and Cos of 33° and some minutes, printed in the first column, below the M. If we go down in that column to 4, meaning an angle of 33°4′, so 33 degrees and 4 minutes (= 4/60 degrees) in that row we read of the values of sin(33°4′) = 0.5456, tan = 0.6511 and cos = 0.8380. These tablebooks were constructed and used by scientists and engineers during the last centuries until computers and calculators came up to give the desired values of the trigonometric functions very fast and accurate.

The best is to set up your calculator in the "radian mode".
Realize that an angle in degrees of $\theta^\circ = 33°4'$ is the same as an angle in radians of

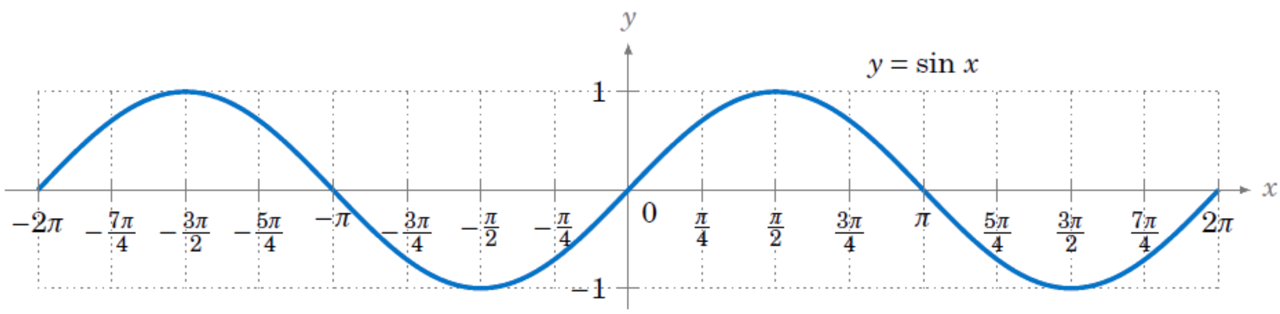
$$\frac{\pi}{180}\times(33+\tfrac{4}{60}) = 0.5771$$

Graphs of the trigonometric functions.

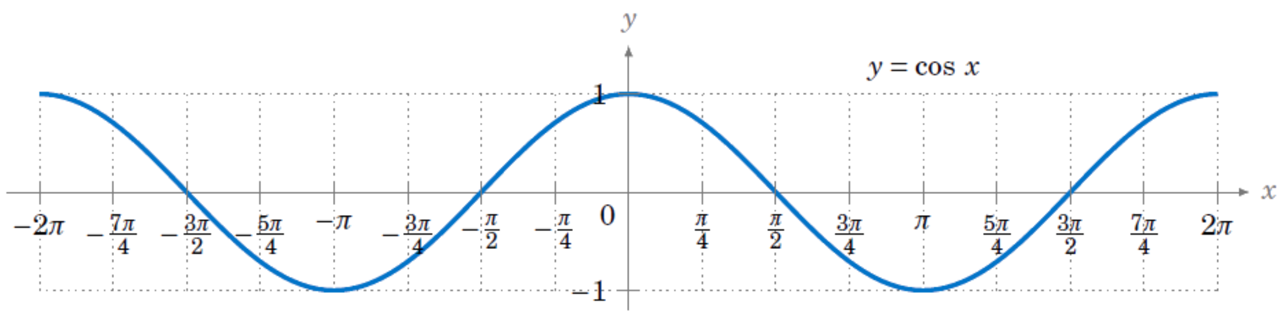
The graph of the <u>sine</u> function is shown below, the independent variable θ expressed in radians. The values of sin(θ) are the Y-coordinates of a point P that goes around the unit circle.



The sine function is a *periodic* function. We call a function a periodic function if there exists a constant $P > 0$ so that $f(x+P) = f(x)$ for all x in the domain of the function. The smallest such P is called the *period* of the function. Because the sine function repeats every 360° = 2π, the period of the sine function is $P = 2\pi$. The graph of the function $y = \sin x$ on the interval $[-2\pi, 2\pi]$ is shown below. It should be emphasized however again that the variable x in that graph is another x as in the figure here above. In fact it is the same as θ. We simply rename θ by x. We even don't think that x is an angle: *it is a real variable*. The graph can be extended infinitely many times in both directions. So the function $f(x) = \sin x$ has as domain: $(-\infty, \infty)$ so all real numbers, and as range: $[-1, 1]$

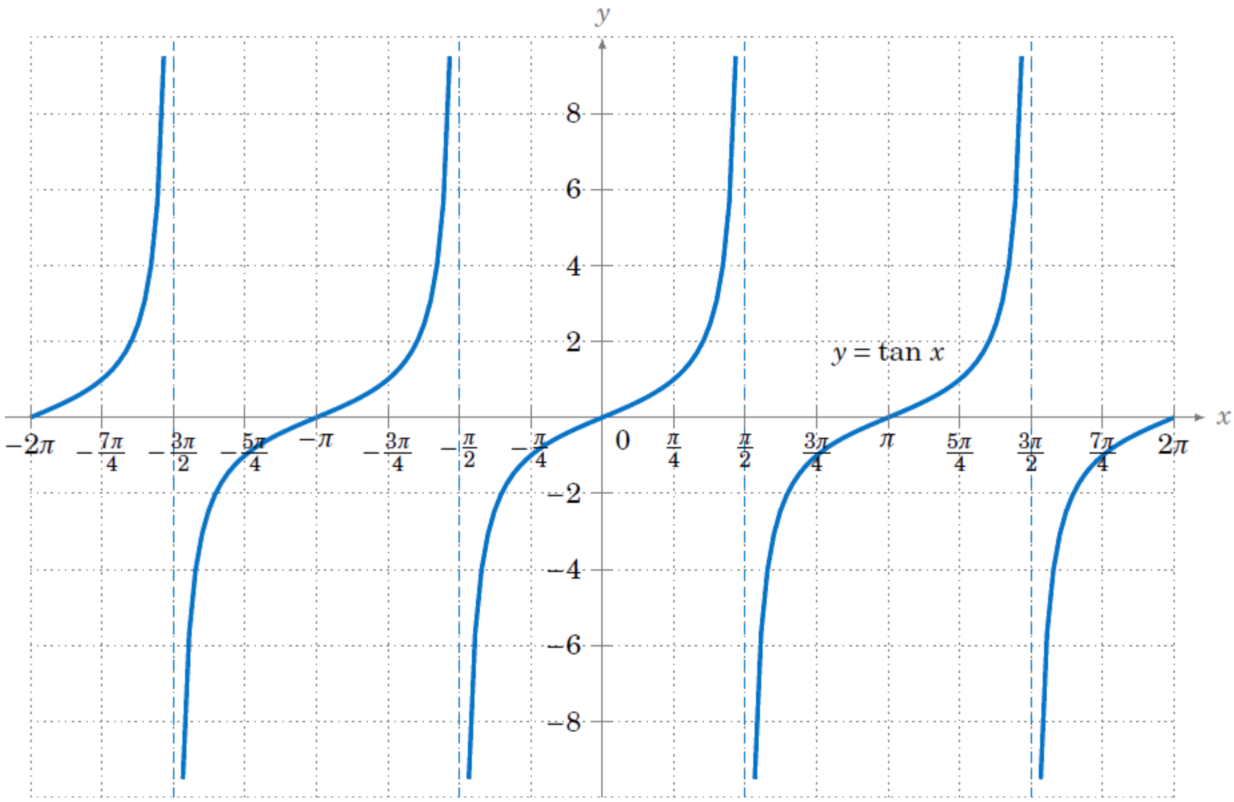


And here the <u>cosine</u> function:



Cosine has the same domain and range as sine. An important relationship is: $\cos x = \sin(x + \frac{1}{2}\pi)$ so $\cos 0$ has the same value as $\sin(\frac{1}{2}\pi)$, $\cos(\frac{1}{2}\pi)$ has the same value as $\sin(\pi)$ and so on. The graph of cos x is the same as the graph of sin x, but shifted $\frac{1}{2}\pi$ to the left.

The function tan x doesn't exist for $x = \frac{1}{2}\pi + k \cdot \pi$ for any integer k. For those values of x the graph has a vertical asymptote.



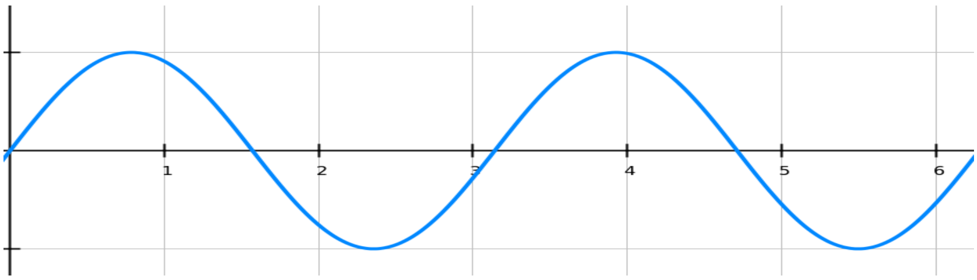
The behaviour of tan x close to the "singular points" can be stated in limits:

$\lim_{x \to \frac{1}{2}\pi -} \tan x = \infty$ and $\lim_{x \to \frac{1}{2}\pi +} \tan x = -\infty$ for e.g. the singular point $x = \frac{1}{2}\pi$

More graphs. Period, frequency and amplitude.

Example: Draw the graph of $f(x) = \sin(2x)$ on the interval $[0, 2\pi]$.
Substituting some simple values give: $f(0) = 0,\ f(\frac{1}{4}\pi) = 1,\ f(\frac{1}{2}\pi) = 0,\ f(\frac{3}{4}\pi) = -1,\ f(\pi) = 0$

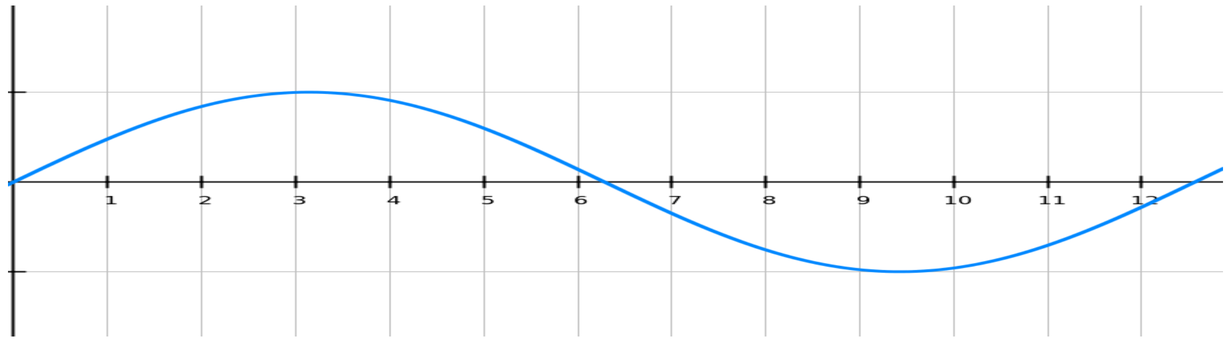
It is clear that if x runs from 0 to 2π then $2x$ takes values from 0 to 4π. And on the interval $[0, 2\pi]$ the graph of $\sin(2x)$ makes two complete "waves".



The period of this function is π.

In general: the function $f(x)=\sin(ax)$, with a is a nonzero constant, has period: $P=\dfrac{2\pi}{a}$

The function $\sin(\frac{1}{2}x)$ has as period $P=\dfrac{2\pi}{1/2}=4\pi$ and graph:

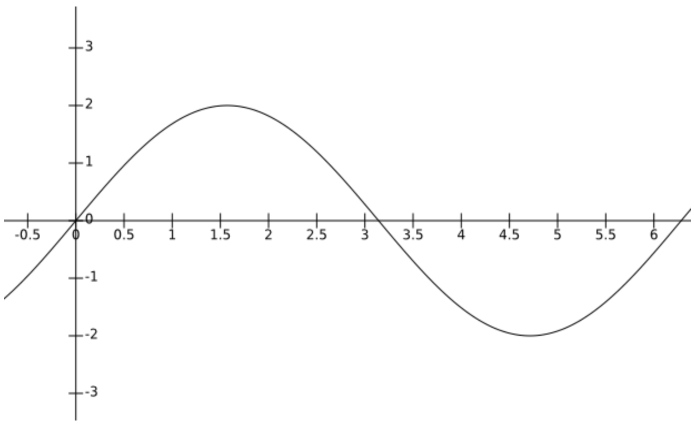


Frequency is a measurement that is closely related to period. In science the frequency of a sound or light wave is the number of complete waves made in a given time period, such as a second. In economic science such a time period can be expressed in years. In trigonometry we measure frequency as the number of complete waves every 2π units. So the frequency of $\sin(2x)$ is 2 and of $\sin(\frac{1}{2}x)$ is ½.

In general: the function $f(x)=\sin(ax)$ has frequency a.

The relation between frequency f and period P is: $f=\dfrac{2\pi}{P}$

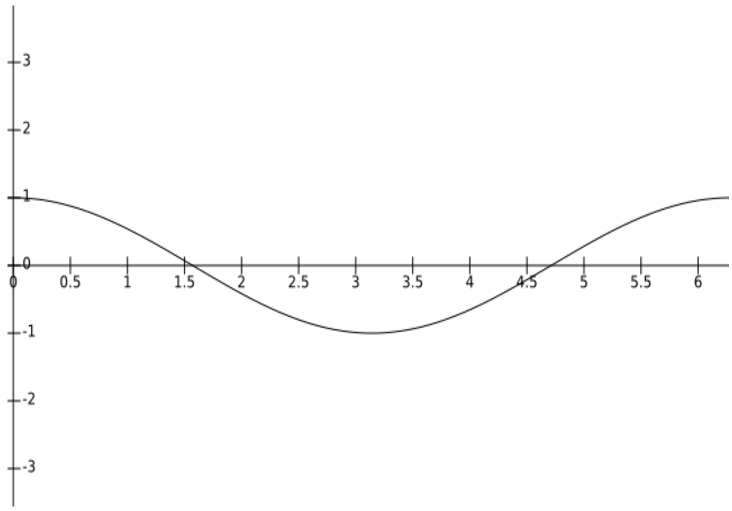
Now we look at the function $f(x)=2\sin x$.



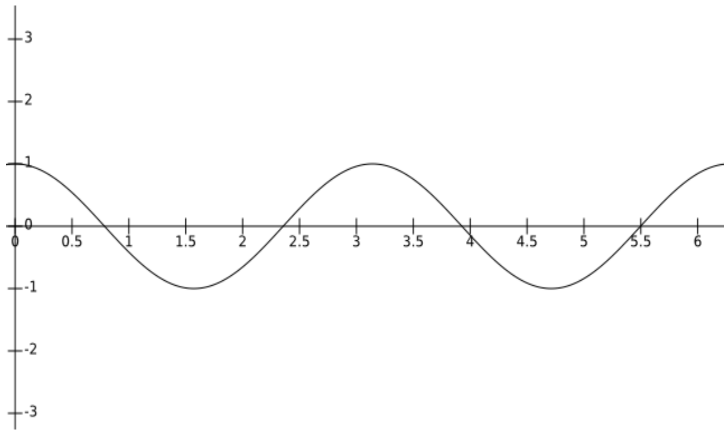
The graph looks similar as the graph of sin x, the same frequency (and period). It has only a wider range: $[-2,2]$ The maximum value is 2 for $x=\frac{1}{2}\pi$ and the minimum value -2 for $x=\frac{3}{2}\pi$.
We call 2 the "amplitude" of the function.
In general: $f(x)=b\sin(ax)$ has amplitude: $|\,b\,|$, the absolute value of b.

Example: draw the graph of $f(x) = -3\cos(2x)$ on $[0, 2\pi]$

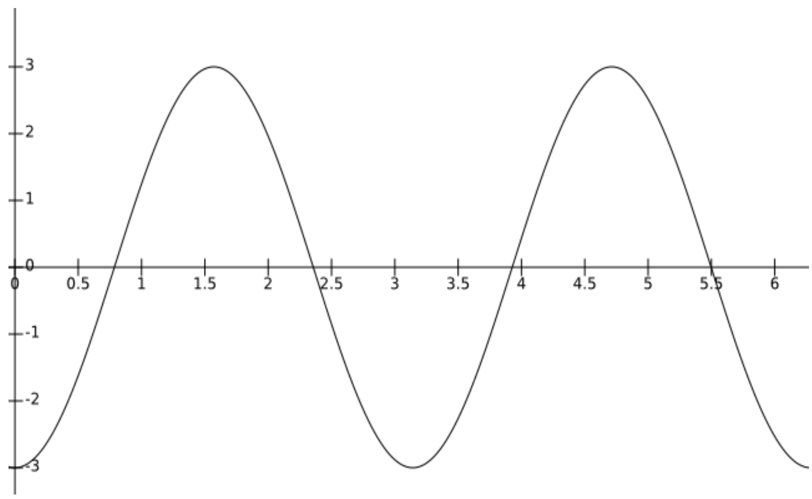
$\cos x$:



$\cos(2x)$: double frequency.

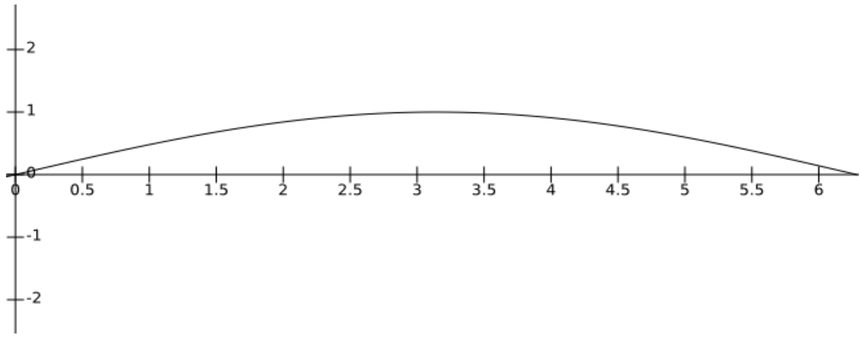


$-3\cos(2x)$: multiply all y-values by -3. The amplitude is 3.

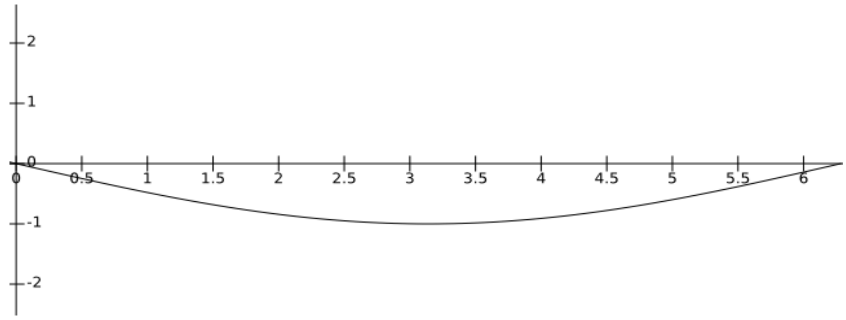


Example: Draw the graph of $f(x) = 1 - \sin(\frac{1}{2}x)$ on $[0, 2\pi]$

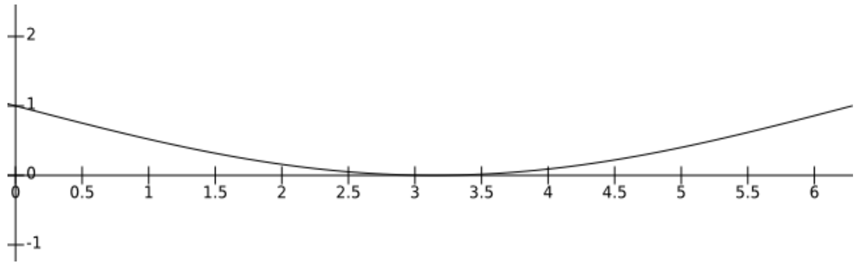
First: $\sin(\frac{1}{2}x)$: period: 4π



Second: $-\sin(\frac{1}{2}x)$: all y-values multiplied by -1.



Third: $1-\sin(\frac{1}{2}x)$: shifted one unit upwards.



Remark that this function has 3 extreme values on the interval $[0, 2\pi]$.
One interior absolute minimum value 0 for $x = \pi$.
Two absolute maximum values 1 at the boundaries, so for $x = 0$ or $x = 2\pi$.

Basic formulas.

We recall that from now on we consider the functions sin x, cos x and tan x as functions of a real variable, sometimes expressed as multiples of π, but sometimes not. The following formulas are immediately clear by the definitions and graphs:

$$\sin(-x) = -\sin x$$
$$\cos(-x) = \cos x$$
$$\tan(-x) = -\tan x$$

This is sometimes expressed by saying that cos x is an "even" function and sin x and tan x are "odd" functions.

Addition formulas are:

$$\sin(x+y)=\sin x\cos y+\cos x\sin y$$
$$\cos(x+y)=\cos x\cos y-\sin x\sin y$$
$$\tan(x+y)=\frac{\tan x+\tan y}{1-\tan x\tan y}$$

We state this formulas without proof. Proofs are not difficult and can be found in elementary textbooks. For $x=y$ we have:

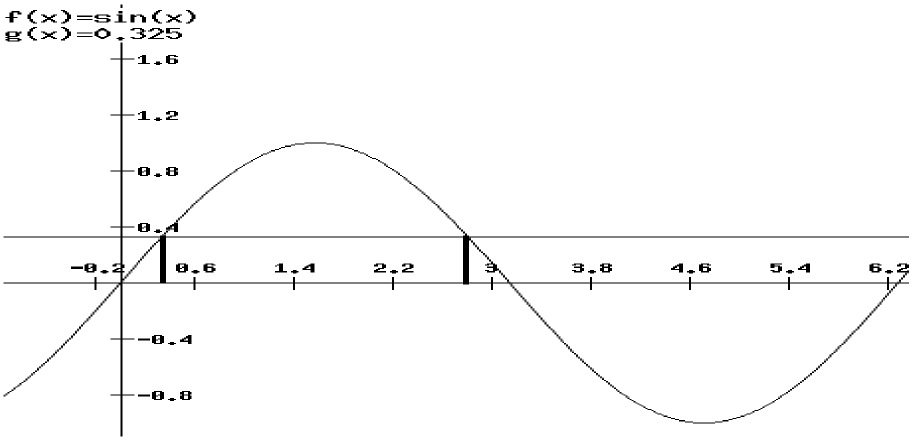
$$\sin(2x)=2\sin x\cos x$$
$$\cos(2x)=\cos^2 x-\sin^2 x$$
$$\tan(2x)=\frac{2\tan x}{1-\tan^2 x}$$

Trigonometric equations:

It is obvious that sin x and cos x are not one-to-one functions. At the contrary: all values between -1 and 1 are assumed infinitely many times. The equation: $\sin x=a$ has no solutions for $a>1$ or $a<-1$ and infinitely many solutions for $-1\le a\le 1$ To solve an equation as $\sin x=0.325$ in the past we used table books. Now we can use a simple pocket calculator (e.g. CASIO *fx* - 85) that gives very fast and accurate the value of x preferably in radians. (Mind the setup of your calculator !) On the interval $-\frac{1}{2}\pi\ \le x\le\frac{1}{2}\pi$ sin x is a monotonic increasing function and on this limited domain sin x is on-to-one. Your calculator gives the unique solution that is lying in this interval, so between $-\frac{1}{2}\pi$ and $\frac{1}{2}\pi$.

With the button SHIFT or INV followed by SIN and then 0.325 the solution is: x = 0.3310 rounded on 4 decimals. What are all the solutions ?

You see that the graph of sin x on $[0,2\pi]$ assumes the value 0.325 (the horizontal line) twice. The smaller value, the value between $-\frac{1}{2}\pi$ and $\frac{1}{2}\pi$, is given by your calculator: 0.3310. The graph of sin x is symmetric around the line $x=\frac{1}{2}\pi$ so the second value of the solutions on $[0,2\pi]$ is: $\pi-0.3310=2.8106$



Because sin x is a periodic function with period 2π all the solutions of the equation $\sin x = 0.325$ are: $x = 0.3310 + k \cdot 2\pi$ or $x = 2.8106 + k \cdot 2\pi$ with k any integer.

If x_1 is a solution of $\sin x = a$ found by a calculator, then $-\frac{1}{2}\pi \le x_1 \le \frac{1}{2}\pi$ and all the solutions are: $x = x_1 + 2k\pi$ or $x = (\pi - x_1) + 2k\pi$ with k any integer.

For some special values for a we don't need a calculator, and we can express the solutions in multiples of π. Example:

Solve: $\sin x = \frac{1}{2}$

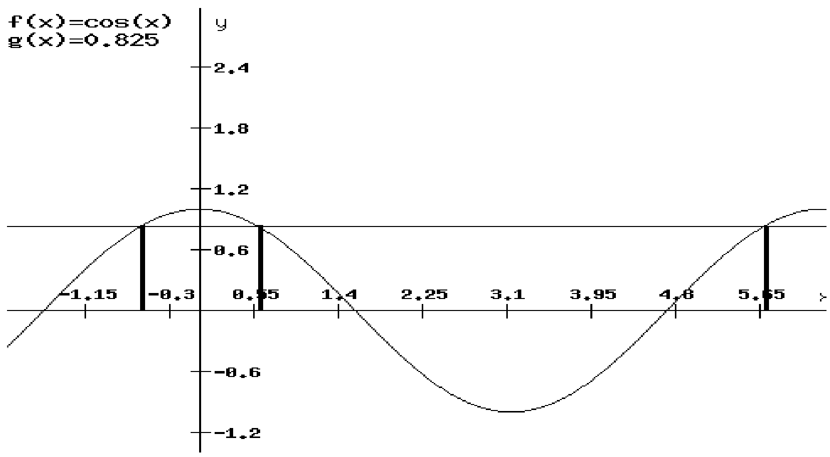
Solution: $x = \frac{1}{6}\pi + 2k\pi \quad \vee \quad x = \frac{5}{6}\pi + 2k\pi$

Now for the cosine:

On the interval $[0, \pi]$ cos x is a monotonic decreasing function from 1 to -1 and on this restricted domain cos x is one-to-one. Your calculator gives the unique solution that is lying in this interval, so between 0 and π, inclusive.

Example: solve $\cos x = 0.825$

With the button SHIFT or INV followed by COS and then 0.825 the solution is: x = 0.6006 rounded on 4 decimals. What are all the solutions ?



You see that the horizontal line $y = 0.825$ crosses the graph of the cosine function three times. One point of intersection has an x-value between 0 and π and is given by your calculator, the already mentioned $x = 0.6006$. The graph is symmetric around the line $x = 0$ (the y-axis) and so another solution of the equation is: − 0.6006. All the solutions are

$$x = 0.6006 + 2k\pi \quad \vee \quad x = -0.6006 + 2k\pi$$

The two solutions in the interval $[0, 2\pi]$ are: $x_1 = 0.6006 \vee x_2 = 5.6826$

For some special values we can express the solutions in multiples of π:

Solve: $\cos x = \frac{1}{2}\sqrt{2}$

solution: $x = \frac{1}{4}\pi + 2k\pi \quad \vee \quad x = -\frac{1}{4}\pi + 2k\pi$

So one of the solutions is: $x = \frac{1}{4}\pi$. But what did we really won by this "solution" ? We have stated a relationship between one irrational , $\frac{1}{2}\sqrt{2}$,and another irrational: $\frac{1}{4}\pi$. But we *may* state: $\cos\frac{1}{4}\pi = \frac{1}{2}\sqrt{2}$ as a *true* equality from a theoretical point of view. Unfortunately π and $\sqrt{2}$ are irrational numbers and cannot be represented by a finite number of digits, so:
cos (0.785398⋯) = 0.707106⋯ is only exactly true if we use an infinite number of digits, which is impossible. In writing: "cos(0.785) = 0.707" we should better use the approximation sign: ≈.

On the interval $[-\frac{1}{2}\pi, \frac{1}{2}\pi]$ the function tan x is monotonic increasing from $-\infty$ to ∞. So the range of the function is: all real numbers. So the equation: $\tan x = a$ has one unique solution, given by your calculator with SHIFT TAN. The function tan x is a periodic function with period $P = \pi$. So if you know the solution x_0 between $-\frac{1}{2}\pi$ and $\frac{1}{2}\pi$ then all solutions are: $x = x_0 + k\pi$

Example: The equation $\tan x = 3.47$ has as all solutions: $x = 1.2902 + k\pi$

Summary:

$$\sin x = \sin a \;\Leftrightarrow\; x = a + 2k\pi \;\vee\; x = (\pi - a) + 2k\pi$$
$$\cos x = \cos a \;\Leftrightarrow\; x = a + 2k\pi \;\vee\; x = -a + 2k\pi$$
$$\tan x = \tan a \;\Leftrightarrow\; x = a + k\pi$$

Trigonometric functions in calculus.

We state without proof: sin x, cos x and tan x are continuous and differentiable functions. We have the following facts (proofs in elementary textbooks):

$$\lim_{x\to 0}\frac{\sin x}{x} = 1 \quad \text{and we may state: } \sin x \approx x \text{ for small values of } x$$
$$\lim_{x\to 0}\frac{1-\cos x}{x} = 0$$

The derivative functions are:

$$\frac{d}{dx}(\sin x) = \cos x$$
$$\frac{d}{dx}(\cos x) = -\sin x$$
$$\frac{d}{dx}(\tan x) = \frac{1}{\cos^2 x}$$

Exercises:

1. Express the following angles in radian measure. You may use π in your answer if desired.

a) 30° d) – 135°
b) 45° e) 290°
c) 1° f) 68° 34'

2. The following angles are expressed in radians. Write those angles in the degree mode.

a) $\frac{1}{3}\pi$ c) 1
b) π d) 2.147

3. Sketch the graphs of the following functions on the interval $[0, 2\pi]$:

a) $f(x) = 1 - 2\sin x$ b) $g(x) = \cos\frac{1}{2}x$ c) $y = \sin(x - \frac{1}{2}\pi)$

4. Solve the next equations on the domain: all real x. You may use π in your answer if desired, or round your answer on 3 decimal places.

a) $\sin x = -\frac{1}{2}$ f) $\sin x = \sqrt{2}$
b) $\cos x = \frac{1}{2}\sqrt{2}$ g) $\sin(0.2x) = 0$
c) $\tan x = \sqrt{3}$
d) $\sin 2x = -1$
e) $\cos x = -0.647$

5. Solve: $3\sin x = 2\cos^2 x$

6. Find the derivatives of the following functions:

$f(x) = 3\sin^2 x$
$g(x) = 3\sin x^2$
$h(x) = 3\sin 3x$
$k(x) = (3\sin 3)x$

7. Differentiate the following functions:

$f(x) = \dfrac{\cos x}{\sqrt{x}}$
$g(x) = 2\sin x \cos x$
$h(x) = x^2 \ln(\cos x)$

8. Let $f(x) = x - 2\sin x$ with domain the closed interval $[0, \pi]$

a) Find all extreme values of the function and classify (maximum / minimum)
b) Show that the graph of the function is convex.
c) Sketch the graph of the function.

9. Find a) $\int \sin x \cdot \cos x \, dx$ b) $\int \sin^2 x \, dx$ c) $\int \sin^3 x \, dx$ d) $\int_0^{2\pi} \sin x \, dx$

10. Let $f(x) = 1 - 2\sin(2x)$ on domain $[0,\pi]$.

a) Solve $f(x) = 0$.

b) Find all extreme values of f.

c) Sketch the graph.

11. Let $f(x) = \frac{1}{2}x + \sin x$ on domain: $0 \le x \le 2\pi$.

a) Find all extreme values of f. Say: local/global, maximum/minimum.

b) Find the inflection points of the graph and find on which part of the domain the graph is convex (concave up).

c) Find the area between the graph of the function and the x - axis.