Repeated Games with Asynchronous Moves*

Quan Wen†

Vanderbilt University

June 2004

Abstract

We study repeated games with asynchronous moves, where not all players may revise their actions in every period. Using a state-dependent backward induction, we derive the infimum of a player's subgame perfect equilibrium payoffs, *the effective minimax value*, in a repeated game. A player's effective minimax value depends on the structure of players' asynchronous moves. We establish an unconventional Folk Theorem: As players become sufficiently patient, any feasible payoff vector that strictly dominate the effective minimax values can be approximated by a subgame perfect equilibrium in the repeated game. This result also implies a number of Folk Theorems and anti-Folk Theorems in the current repeated game literature.

JEL Classification: C72, C73

Keywords: Folk Theorem, repeated games, asynchronous moves, effective minimax, maximin value

---

*I would like to thank Dilip Abreu, Taiji Furusawa, David Frankel, Michihiro Kandori, Roger Lagunoff, Eric Maskin, Akihiko Matsui, Philip Reny, Jeroen Swinkels, Satoru Takahashi, and Kiho Yoon for their comments and suggestions. The usual disclaimer applies.

†Address: Department of Economics, Vanderbilt University, VU Station B #351819, 2301 Vanderbilt Place, Nashville, TN 37235-351819 U.S.A. Phone (615) 322-0174, Fax: (615) 343-8495, Email: quan.wen@vanderbilt.edu.

1 Introduction

Players behave differently in an one-shot game and in the corresponding repeated game due to their different objectives in short-run and long-run. When players evaluate their future high enough, a repeated game admits almost all "reasonable" outcomes in subgame perfect equilibrium, a result known as the Folk Theorem.[1] In one seminal paper, Fudenberg and Maskin (1986) establish the following Folk Theorem for infinitely repeated games with discounting: Under certain conditions, any feasible and strictly individually rational payoff vector of a normal-form stage game can be supported by subgame perfect equilibrium in the corresponding repeated game. Players play the stage simultaneously and synchronously in every period of the repeated game. Fudenberg and Tirole (1991) first raise the issues on repeated games when players do not revise their actions simultaneously and synchronously. For repeated games with non-simultaneous moves, Rubinstein and Wolinsky (1995), Sorin (1995), and Wen (2002) take a direct approach by studying infinitely repeated extensive-form games and find that under the full-dimensionality condition (even for two-player games), the conventional Folk Theorem holds for the repeated normal-form representation of the extensive-form stage game. Without imposing any dimensionality condition on a sequential stage game, Wen (2002) introduces players' effective minimax values and establishes an unconventional Folk Theorem for the corresponding repeated sequential game.[2]

In this paper, we study infinitely repeated games with asynchronous moves, where not all players may revise their actions in every period. By taking the dynamic effects of players' actions in the stage game and the idea of effective minimax into account, we derive the infimum of a player's subgame perfect equilibrium payoffs (the effective minimax value) in a repeated game. Depending on players' asynchronous move structure, a player's

[1] See Friedman (1971), Aumann (1981), Fudenberg and Tirole (1991), and Benoit and Krishna (1998).

[2] The concept of an effective minimax value is first introduced in Wen (1994) for normal-form games and is generalized in Takahashi (2003) to extensive-form games with almost perfect information.

effective minimax value in a repeated game generally differs from his effective minimax value in the underlying stage game. We establish an unconventional Folk Theorem in the sense that the limiting set of subgame perfect equilibrium payoffs in a repeated game is bounded below by players' effective minimax values in the repeated game, rather than by their effective minimax values in the underlying stage game. Our main result asserts that when players become sufficiently patient, any feasible payoff vector that strictly dominates players' effective minimax values can be approximated by subgame perfect equilibrium in the repeated game. Results in this paper fully characterize the limiting set of subgame perfect equilibrium payoffs since a player receives no less than his effective minimax value in any subgame perfect equilibrium. This paper contributes to the current repeated game literature in many dimensions. Our results not only establishes a formal linkage between the limiting set of subgame perfect equilibrium payoffs and the asynchronous move structure of the repeated game, but also integrates a number of Folk Theorems and anti-Folk Theorems in the current literature.

Since Lagunoff and Matsui (1997) first demonstrate a unique subgame perfect equilibrium in the infinitely repeated pure coordination game with alternating moves, many researchers have studied repeated games with asynchronous moves.[3] The violation of the full-dimensionality condition in the pure coordination game alone cannot explain Lagunoff and Matsui's (1997) anti-Folk Theorem results since the conventional Folk Theorem does not require the full dimensionality in two-player repeated games. The structure of players' asynchronous moves plays a crucial role in determining the limiting set of subgame perfect equilibrium payoffs. Yoon (2001) studies asynchronously repeated games under the assumption that the players who cannot revise their actions will have to play the same actions (including mixed actions) from past, and obtains the conventional Folk Theorem under the NEU (Non-Equivalent Utility) and the FPI (Finite Period Inaction) conditions.

[3]See Lagunoff and Matsui (2001), Haller and Lagunoff (2000b), Yoon (2001, 2004), and Takahashi and Wen (2003).

Takahashi and Wen (2003) demonstrate, however, the conventional Folk Theorem may not fully characterize the limiting set of subgame perfect equilibrium payoffs in asynchronously repeated games. In our model, although the players who cannot revised their actions will have to play the realization of their past mixed actions, we show that the set of feasible and individually rational payoffs is a subset subgame perfect equilibrium payoffs in an asynchronously repeated game. A repeated game with asynchronous moves can be considered a stochastic game studied by Dutta (1995), Haller and Lagunoff (2000a), and many others. Under the full-dimensionality condition, Dutta (1995) obtains a Folk Theorem under players' minimax values calculated from equilibrium strategies from the entire stochastic game. Dutta's (1995) result also suggests that the conventional Folk Theorem may not be valid to characterize the limiting set of subgame perfect equilibrium payoffs in stochastic games. Our results strengthen his finding by establishing a formal linkage between the limiting set of subgame perfect equilibrium payoffs and the underlying dynamic structure.

One can analyze some special repeated games with asynchronous moves using the technique developed for repeated sequential games. Consider a repeated game of the following battle of the sexes (BOS) game:

$1 \setminus 2$	L	R
U	2, 1	0, 0
D	0, 0	1, 2

where player 1 moves in every period and player 2 moves once in every other period. Let $\delta \in (0, 1)$ be players' common discount factor per period. Note that every player's (mixed) minimax value is $2/3$. The game played over the two consecutive periods in which player 1 moves in both periods and player 2 moves only in the first period can be considered as "two-step" sequential game. Such a "two-step" sequential game, of course, depends on players' discount factor $\delta \in (0, 1)$.

Now consider this "two-step" sequential game. Note that player 2's payoff in the second step can be as low as 0 (player 1 chooses U if player 2 plays R in the first step, and

D if player 2 plays L in the first step), and player 2's payoff in the first step can be as low as his minimax value $2/3$. Player 2's lowest possible subgame perfect equilibrium payoff over the two steps (periods) is $2/3$, which is approximately $1/3$ per period when δ is large enough. On the other hand, since player 1's payoff in the second step cannot be lower than 1 (player 1 chooses D if player 2 plays R in the first step, and U if player 2 plays L in the first step), player 1's lowest possible subgame perfect equilibrium payoff over the two steps is not less than $(2+3\delta)/3(=2/3+\delta)$, which is approximately $5/6$ per period when δ is large enough. These simple observations are inconsistent with the conventional Folk Theorem.[4] Wen's (2002) unconventional Folk Theorem is governed by players' effective minimax values (the minimax values in this case) in the normal-form representation of the "two-step" sequential game. It also demonstrates how players' effective minimax values in a sequential game may explain Lagunoff and Matsui's (1997) anti-Folk Theorem. It is important to point out here that the model of repeated sequential games is, however, inadequate to deal with repeated games with asynchronous moves. The study of repeated sequential games inspires a state-dependent backward induction technique developed here to analyze repeated games with asynchronous moves. It takes the dynamic effects of players' actions into account in formulating player's effective minimax values. Dealing with the stage game as a static game will typically ignore these dynamic effects, and so the conventional Folk Theorem can be misleading. Results from this paper assert not only what can be supported by subgame perfect equilibrium, but also, and more importantly, what cannot be supported by subgame perfect equilibrium in a repeated game with asynchronous moves.

The rest of this paper is organized as follows. In Section 2, we set up the model of repeated games with asynchronous moves and review the concept of effective minimax

[4]If player 1 moves in every period while player 2 moves once every T periods, then player 1's lowest possible subgame perfect equilibrium payoff cannot be less than $(T-1+2/3)/T$, while player 2's lowest possible subgame perfect equilibrium payoff can be sufficiently close to $(T-1+2/3)/(2T)$, when δ is large enough. Note that by the feasibility, player 2 must receive at least half of what player 1 receives in this repeated game.

values for normal-form games. In Section 3, we present two examples to illustrate a state-dependent backward induction in order to derive the infimum of a player's subgame perfect equilibrium payoffs. In Section 4, we formally derive and study a player's effective minimax value in a repeated game with asynchronous moves. Section 5 contains our main result, an unconventional Folk Theorem for repeated games with asynchronous moves. Section 6 offers a few concluding remarks.

2 Preliminaries

Consider a finite normal-form game $G = \{A_i, u_i(\cdot);\ i \in I\}$, where I is the set of n players, A_i is the set of player i's *pure actions*, and $u_i(\cdot) : \times_{j\in I} A_j \to \mathbf{R}$ is player i's *payoff function* for all $i \in I$. For convenience, denote $A \equiv \times_{j\in I} A_j$ and $u(\cdot) \equiv (u_1(\cdot), \cdots, u_n(\cdot))$. Let Σ_i be the set of player i's *mixed actions* and $\sigma_i \in \Sigma_i$ be a generic mixed action of player i. With slightly abuse of notation, let $u_i(\cdot)$ also denote player i's *expected payoff function* from a mixed action profile $\sigma = (\sigma_1, \cdots, \sigma_n) \in \Sigma = \times_{j\in I}\Sigma_j$. The *set of feasible payoffs* is the convex hull of $u(A)$, $F = Co[u(A)] \subset \mathbf{R}^n$. By convention, we decompose a pure action profile as $a = (a_i, a_{-i}) \in A$ and a mixed action profile as $\sigma = (\sigma_i, \sigma_{-i}) \in \Sigma$ for all $i \in I$. Player i's standard mixed *minimax value* in the normal-form game G is defined as[5]

$$m_i^s = \min_{\sigma_{-i}\in\Sigma_{-i}} \max_{a_i\in A_i} \ u_i(a_i, \sigma_{-i}). \tag{1}$$

A payoff vector that strictly (weakly) dominates $m^s = (m_1^s, \ldots, m_n^s)$ is strictly (weakly) *individually rational*. The set of feasible and strictly individually rational payoffs in the normal-form game G is then

$$F^* = \{(v_1, v_2, \cdots, v_n) \in F|\ v_i > m_i^s,\ \forall\ i \in I\}. \tag{2}$$

[5]It does not lose any generality by restricting player i's actions within his pure actions. Player i's pure minimax value $\min_{a_{-i}} \max_{a_i} u_i(a_i, a_{-i})$ is defined by restricting other players' actions within their pure actions.

Fudenberg and Maskin (1986) prove the following Folk Theorem for $G^\infty(\delta)$, the infinitely repeated game of G with a discount factor $\delta \in (0, 1)$ per period:

Folk Theorem: (Fudenberg and Maskin, 1986) *If $n = 2$ or F^* has a dimension of n (the full dimensionality condition), then every $v \in F^*$ can be supported by subgame perfect equilibrium in $G^\infty(\delta)$ for a large enough $\delta \in (0, 1)$.*

Fudenberg and Maskin's Folk Theorem directly asserts what can be supported by subgame perfect equilibrium in a repeated game. Another aspect of this result, which is probably even more important, is that any subgame perfect equilibrium payoff vector must be, at least, weekly individually rational.

In a repeated game of G with asynchronous moves, not all players can revise their actions in every period. Let $I_t \subseteq I$ denote the set of players who may revise their actions in period $t \geq 0$. It is assumed that all players choose their actions at the beginning of the repeated game (period 0) so $I_0 = I$. Accordingly, the players who are not in I_t cannot change their actions in period t. Such a repeated game model consists of the normal-form stage game G, a common discount factor $\delta \in (0, 1)$, and the structure of asynchronous moves $\mathcal{I} = \{I_t\}_{t=0}^\infty$ with $I_0 = I$. So-defined repeated games with asynchronous moves, denote as $G^\infty(\delta, \mathcal{I})$, includes many existing important repeated game models as special cases. For example, $G^\infty(\delta, \mathcal{I})$ reduces to a conventional repeated normal-form game (Fudenberg and Maskin, 1986) when $I_t = I$ for all $t \geq 0$. It is a repeated game with alternating moves (Lagunoff and Matsui, 1997) if $I_t = \{1\}$ for all odd $t > 0$ and $I_t = \{2\}$ for all even $t > 0$. It also generalizes asynchronously repeated games (Yoon, 2001) and repeated sequential games (Wen, 2002).

A player may play a mixed action whenever he revises his action. We assume that all past mixed actions are publicly observable. A history at t consists of all past actions, $h_t = (\sigma^0, \sigma^1, \ldots, \sigma^{t-1})$, such that $\sigma^0 \in \Sigma$ and

$$\sigma_i^s \text{ is a realization of } \sigma_i^{s-1} \text{ for } i \notin I_s \text{ and } 0 < s < t. \tag{3}$$

Condition (3) ensures the consistency under the given asynchronous move structure $\mathcal{I}$. It means that the players who cannot revise their actions will have to play the realizations of their past actions. Histories are used to identify a deviator so that the other players can coordinate strategies to punish the deviator in the continuation game. The set of histories in period t is denoted as H_t, while H_0 contains only the null history $\emptyset$ at the beginning of $G^\infty(\delta, \mathcal{I})$.

A strategy profile specifies stage game actions for those in I_t after any history h_t in period t. Since player i cannot revise his actions in period t for $i \notin I_t$, it is unnecessary to specify player i's action in period t. A player i's strategy is a function that maps from all relevant histories for player i into the set of his mixed actions, $f_i : \bigcup_{i \in I_t} H_t \to \Sigma_i$. A strategy profile $f = (f_1, f_2, \cdots, f_n)$ induces a unique probability distribution on the set of pure outcome paths of $G^\infty(\delta, \mathcal{I})$. A pure outcome path $\pi = (a^0, a^1, \cdots, a^t, \cdots)$ satisfies the consistency imposed by $\mathcal{I}$, i.e., $a_i^t = a_i^{t-1}$ for all t and $i \notin I_t$. Players' (average discounted) payoffs from the outcome path π are given by

$$u(\pi) = (1 - \delta) \sum_{t=0}^{\infty} \delta^t u(a^t). \tag{4}$$

Every player evaluates his strategies based on his expected payoff calculated from the induced probability distribution over the set of pure outcome paths. In this paper, we consider subgame perfect equilibrium, which induces a Nash equilibrium in every subgame of $G^\infty(\delta, \mathcal{I})$.

In the rest of this section, we review the concept of effective minimax values in normal-form games. According to Abreu, Dutta and Smith (1994), players i and j have equivalent utilities if $\exists\ \alpha_{ij} > 0$ and β_{ij}, such that

$$u_j(\sigma) \ = \ \alpha_{ji} \cdot u_i(\sigma) + \beta_{ji}, \quad \forall\ \sigma \in \Sigma. \tag{5}$$

Definition 1 *The normal-form game G satisfies the NEU (Non-Equivalent Utility) condition if no two players have equivalent utilities, i.e., $\nexists\ i, j \in I$ such that (5) holds.*

$\forall\ i \in I$, let $E_i \subset I$ be the set of the players (including player i) who have equivalent utilities with player i. By definition, $E_i = E_j$ if and only if players i and j have equivalent utilities. The NEU condition require that $E_i = \{i\}$ for all $i \in I$. Without this NEU condition, the players in E_i have incentive not to minimize player i's payoff after player i deviates in a repeated game. Based on this idea, player i's *effective minimax value* in G is defined as

$$m_i^e = \min_{\sigma} \left[\max_{j \in E_i} \max_{a_j} \quad u_i(\sigma) \right], \tag{6}$$

which is the minimum of player i's payoff under the best unilateral deviation by those who have equivalent utilities with player i. Under the NEU condition, it is obvious that $m_i^e = m_i^s$ due to the fact that $E_i = \{i\}$ for all $i \in I$. It is straightforward that $m_i^e \geq m_i^s$ for all $i \in I$. From any action profile that solves (6), player i's payoff is generally less than m_i^e. Under the best unilateral deviation by those in E_i from a solution to (6), player i obtains his effective minimax value m_i^e.

Any payoff vector that strictly (weakly) dominates $m^e \equiv (m_1^e, \cdots, m_n^e)$ is strictly (weakly) *EUC* (Equivalent Utility Class) rational. Without imposing any dimensionality condition on the stage game G, Wen (1994) obtains the following unconventional Folk Theorem for $G^\infty(\delta)$: Any feasible and strictly EUC rational payoff vector can be supported by subgame perfect equilibrium in $G^\infty(\delta)$ with a large enough $\delta \in (0, 1)$. Any subgame perfect equilibrium payoff vector must be, at least, weakly EUC rational. A player's effective minimax is indeed the infimum of the player's subgame perfect equilibrium payoffs in $G^\infty(\delta)$ as δ goes to 1.

3 Two Examples

We now present two examples to illustrate the infimum of a player's subgame perfect equilibrium payoffs (effective minimax value) in repeated games with asynchronous moves.

The underlying stage game in the first example satisfies the NEU condition, while the one in the second example does not.

3.1 The Battle of the Sexes

Reconsider the battle of the sexes (BOS) game. Recall that there are two pure strategy Nash equilibria, (U, L) and (D, R), and one mixed strategy Nash equilibrium in which player 1 plays U with probability 2/3 and player 2 plays L with probability 1/3. Two players have a common mixed minimax value of 2/3.

Consider a repeated BOS game with alternating moves. Two players move simultaneously in period 0, player 1 moves in all odd periods, and player 2 moves in all even periods. That is, $I_0 = I$ by default, $I_t = \{1\}$ for all odd t, and $I_t = \{2\}$ for all even t. As the alternating-offer bargaining game of Rubinstein (1982) and repeated pure coordination game with alternating moves of Lagunoff and Matsui (1997), this repeated BOS game is structurally cyclical for every two periods (except period 0). With a state-dependent backward induction technique similar to that of Shaked and Sutton (1984), we now derive the infimum of a player's subgame perfect equilibrium payoffs.

The continuation game in any odd period t obviously depends on player 2's action from period $t - 1$, which is referred as the *initial state* of the continuation game. Denote the infimum of player 1's subgame perfect equilibrium payoffs as $m_1(a_2)$ for the initial state $a_2 \in \{L, R\}$. Due to the cyclical structure, player 1's subgame perfect equilibrium payoffs in period $t+2$ are not less than $m_1(a_2')$ if player 2 plays a_2' in period $t+2$. Player 1's equilibrium payoffs from playing $a_1 \in \{U, D\}$ in period t are not less than

$$m_1(a_1|a_2) = (1-\delta)u_1(a_1, a_2) + \delta \min_{\sigma_2 \in \Sigma_2} \left[(1-\delta)u_1(a_1, \sigma_2) + \delta m_1(\sigma_2)\right], \tag{7}$$

The first term on the right hand side of (7) represents player 1's payoff during period t and $m_1(\sigma_2)$ in the second term represents the expected value of player 1's lowest possible continuation payoff from $t + 2$ onward under player 2's mixed action σ_2 in period $t + 1$.

Since player 1 may choose an action to maximize his continuation payoff (7) in period t, we have

$$m_1(a_2) = \max_{\sigma_1 \in \Sigma_1} m_1(\sigma_1|a_2), \tag{8}$$

where $m_1(\sigma_1|a_2) = E_{\sigma_1} m_1(a_1|a_2)$ represents the expected value of $m(a_1|a_2)$ under player 1's mixed action σ_1 in period t. For the BOS game, it is tedious, but nevertheless straightforward, to show that (7) and (8) yield[6]

$$m_1(a_2) = \begin{cases} \frac{(1-\delta)(2+\delta^2)}{1-\delta^4} & \text{if } a_2 = L, \\ \frac{(1-\delta)(1+2\delta^2)}{1-\delta^4} & \text{if } a_2 = R. \end{cases} \tag{9}$$

Notice that $m_1(L) > m_1(R) > 3/4 > 2/3$ for all $\delta \in (0, 1)$, and both $m_1(L)$ and $m_1(R)$ decrease and converge to $3/4$ as δ goes to 1. This means that player 1's continuation equilibrium payoffs in any odd period are not less than $3/4$ as δ goes to 1. Consequently, player 1's subgame perfect equilibrium payoffs in period $t = 0$ are not less than

$$m_1(\emptyset) = \min_{\sigma_2} \max_{\sigma_1} [(1-\delta)u_1(\sigma_1, \sigma_2) + \delta m_1(\sigma_2)] = \frac{(1-\delta)(2+4\delta+\delta^3+2\delta^4)}{3(1-\delta^4)}, \tag{10}$$

which also converges to $3/4$ as δ goes to 1. This implies that player 1's equilibrium payoffs are not less than $3/4$ when the two players become sufficiently patient. In order to achieve the corresponding values in (9) and (10), the two players begin with the mixed strategy Nash equilibrium and then follow a path with a four-period cycle, depending the realization of player 2's mixed action. For example, if player 2 plays L in period 0 then the two players will play a path with the following cycle of actions starting from period 1:

$$(U, L) \Rightarrow (U, R) \Rightarrow (D, R) \Rightarrow (D, L) \Rightarrow (U,\ L).$$

During such a path, player 1 always chooses the action to maximize his stage game payoff and player 2 always chooses the action to minimize player 1's stage game payoff.

[6] Detailed calculation is available upon request.

Accordingly, player 1 receives 2, 0, 1 and 0 over a typical four-period cycle, that is a total payoff of 3 over 4 periods. As δ goes to 1, player 1 receives roughly 3/4 per period from any period with any initial state.

By symmetry, player 2's continuation payoffs in an even period with the initial state $a_1 \in \{U, D\}$ are not less than

$$m_2(a_1) = \begin{cases} \frac{(1-\delta)(1+2\delta^2)}{1-\delta^4} & \text{if } a_1 = U, \\ \frac{(1-\delta)(2+\delta^2)}{1-\delta^4} & \text{if } a_1 = D. \end{cases}$$

In an odd period with an initial state a_2, even if player 1 chooses an action to minimize player 2's continuation payoff, player 2's continuation payoffs are not less than

$$m_2'(a_2) = \min_{a_1}[(1-\delta)u_2(a_1, a_2) + \delta m_2(a_1)] = \begin{cases} \frac{\delta(1-\delta)(2+\delta^2)}{1-\delta^4} & \text{if } a_2 = L, \\ \frac{\delta(1-\delta)(1+2\delta^2)}{1-\delta^4} & \text{if } a_2 = R. \end{cases} \tag{11}$$

At the beginning of this repeated game, player 2's equilibrium payoffs are not less than the minimax value of player 2's continuation payoffs,

$$m_2(\emptyset) = \min_{\sigma_1} \max_{\sigma_2} \Big[(1-\delta)u_2(\sigma_1, \sigma_2) + \delta m_2'(\sigma_1) \Big] = \frac{(1-\delta)(2+\delta^2)(1+2\delta^2)}{3(1-\delta^4)}. \tag{12}$$

In order to achieve these values from (11) and (22), player 1 plays U with probability $(2+\delta^2)/(3+3\delta^2)$ so player 2 is indifferent between his two actions in period 0. After period 0, two players follow the path of a four-period cycle that reverses the one in obtaining $m_1(\emptyset)$. Again, player 2 receives 3 out of 4 periods, which is why that both (11) and (12) also converge to 3/4 as δ goes to 1.

Results we have so far imply that any equilibrium payoff vector in this repeated BOS game must weakly dominate (3/4,3/4) as δ goes to 1, which is very different from what the conventional Folk Theorem predicts. What we learn from this example is that, even under the full dimensionality condition, the asynchronous move structure plays a crucial role in determining the limiting set of subgame perfect equilibrium payoffs in a repeated game.

3.2 The Pure Coordination

Next we consider the following pure coordination (PC) game in which $a > \max\{b, c, d\}$:

$1 \setminus 2$	L	R
U	$a,\ a$	$b,\ b$
D	$c,\ c$	$d,\ d$

The two players have equivalent utilities and $m_1^e = m_2^e = \max\{b, c, d\}$. Lagunoff and Matsui (1997) study the repeated PC game with alternating moves and show that the repetition of (U, L) (with a payoff (a, a)) is the unique subgame perfect equilibrium for all $\delta \in (0, 1)$.

Here we reconsider this game in order to demonstrate how our state-dependent backward induction works when the NEU condition fails. Since the two players have equivalent utilities, player 2 would maximize (rather than minimize) player 1's payoff in player 1's worst subgame perfect equilibrium (which is also player 2's worst subgame perfect equilibrium). This changes the nature of the optimization problem (7). Adopting the same set of notations, now we have

$$m_1(a_2) \quad = \quad \max_{a_1} \left[(1-\delta)u_1(a_1, a_2) + \delta \max_{a_2'} m_1(a_2') \right]. \tag{13}$$

It is obvious that $m_1(L) = a$ from (13). With $m_1(L) = a$, (13) implies that

$$m_1(R) \geq m_1(U|R) \geq (1-\delta)b + \delta a \to a, \quad \text{as } \delta \to 1.$$

Since $m_1(R) \leq a$ from the set up, $m_1(R)$ converges to a as δ goes to 1. In period 0, player 1's equilibrium payoffs are not less than

$$m_1(\emptyset) \ = \ \max_{a_2} \ \max_{a_1} [(1-\delta)u_1(a_1, a_2) + \delta m_1(a_2)] \ = \ a.$$

Therefore, we obtain Lagunoff and Matsui's (1997) results: The (continuation) equilibrium payoff vector is (a, a) in period 0, any odd period with initial state L, and any even period with initial state U, or is arbitrarily close to (a, a) in any period when the two players are sufficiently patient.

4 Effective Minimax

In the previous section, we demonstrate how to derive a lower bound of player's subgame perfect equilibrium payoffs in repeated games with and without the NEU condition. Now we consider this state-dependent backward induction in $G^{\infty}(\delta, \mathcal{I})$. To do that, we denote $\sigma_{I'}$ and $a_{I'}$ as a mixed and a pure action profiles by the players who are in I', and denote $-I'$ as the complement of I' for all $I' \subseteq I$.

Any pure-action profile by the players who do not revise their actions in period t, denoted as $a^t_{-I_t}$, is an *initial state* of the continuation game in period t since their actions are fixed in period t. The initial state is denoted as $\emptyset$ if $I_t = I$ (such as in period 0). For any $t \geq 0$ and $i \in I$, denote $m^t_i(a_{-I_t})$, player i' *effective minimax value*, as player i's lowest subgame perfect equilibrium payoff in the continuation game in period t with an initial state a_{-I_t}.

In any subgame perfect equilibrium, player i's continuation payoffs in period t are not less than the lowest value of a weighted sum of player i's payoff during period t and his corresponding effective minimax value in period $t+1$. Player i's payoff in period t and his effective minimax value in period $t+1$ depends on the initial state in period t and what the players play during period t. Different actions in period t lead to different payoffs to player i during period t and different initial states in period $t+1$. Recall that the players in I_t move simultaneously. By properties of effective minimax in a normal-form game, player i's continuation payoffs in period t are not less than the effective minimax value of player i's continuation payoffs in period t:

$$m^t_i(a_{-I_t}) = \min_{\sigma_{I_t} \in \Sigma_{I_t}} \max_{j \in E_i \cap I_t} \max_{a_j \in A_j} \Big[(1-\delta) u_i(\sigma_{I_t}, a_{-I_t}) + \\ \delta E_{\sigma_{I_t}} m^{t+1}_i (a'_{(-I_{t+1}) \cap I_t}, a_{(-I_{t+1}) \cap (-I_t)}) \Big]. \tag{14}$$

The second term on the right hand side of (14) is the expected value of player i's effective minimax value in period $t+1$ from the probability distribution on the initial states in period $t+1$ induced by σ_{I_t}. $a'_{(-I_{t+1})\cap I_t}$ in (14) denotes the realization of $\sigma_{(-I_{t+1})\cap I_t}$. Different from (6), the objective function in (14) is player i's lowest continuation payoff in period t. The players who revise their actions in period t effectively minimize player i's lowest continuation payoff, under the best unilateral deviation by any player who has equivalent utility with player i and also revise their actions in period t. The actions played in period t influences not only player i's payoff during period t, but also the initial state in period $t+1$, and hence player i's effective minimax value in period $t+1$. Equation (14) captures the dynamic effects of players' actions played in period t. From the definition of a player's effective minimax value in $G^\infty(\delta, \mathcal{I})$, the following Theorem 1 is immediate:

Theorem 1 *In $G^\infty(\delta, \mathcal{I})$, $\forall\ \delta \in (0,1)$, $t \geq 0$ and $a_{-I_t} \in A_{-I_t}$, player i's continuation equilibrium payoffs in period t with initial state a_{-I_t} are not less than $m_i^t(a_{-I_t})$, defined recursively by (14).*

It is easy to demonstrate that $m_i^t(a_{-I_t})$ reduces to m_i^e in conventional repeated games in which there is only an initial state $\emptyset$ in every period since $I_t = I$ for all t. Accordingly, (14) implies that

$$\begin{aligned} m_i^t(\emptyset) &= \min_{\sigma\in\Sigma} \max_{j\in E_i\cap I} \max_{a_j\in A_j} \left[(1-\delta)u_i(\sigma) + \delta E_{\sigma_{I_t}} m_i^{t+1}(\emptyset) \right], \\ &= (1-\delta) \min_{\sigma\in\Sigma} \max_{j\in E_i\cap I} \max_{a_j\in A_j} u_i(\sigma) + \delta m_i^t(\emptyset), \\ \Rightarrow \quad m_i^t(\emptyset) &= \min_{\sigma\in\Sigma} \max_{j\in E_i\cap I} \max_{a_j\in A_j} u_i(\sigma) \quad = \quad m_i^e. \end{aligned}$$

When $I_t = I_{t+T}$ for all $t \geq 0$ and some finite T, $G^\infty(\delta, \mathcal{I})$ can be treated as a repeated sequential game, where the game played in the first T periods of $G^\infty(\delta, \mathcal{I})$ is a sequential stage game with T steps. There is only one initial state $\emptyset$ in periods kT for all non-negative integer k and so $m_i^0(\emptyset) = m_i^T(\emptyset)$. Apply (14) for the first T periods yield the average value of player i's effective minimax value in the sequential stage game.[7] For a repeated game

[7]Note that the class of repeated games with asynchronous moves is neither a superset nor a subset of the class of repeated sequential games.

with cyclic moves (such as alternating moves), $I_t = I_{t+T}$ for all $t > 1$ and some finite $T > 0$, periods t and $t + T$ have the same set of initial states. $m_i^t(a_{-I_t}) = m_i^{t+T}(a_{-I_t})$ for all a_{-I_t} can be solved by recursive substitution of (14) over T consecutive periods.

Under the NEU condition, $m_i^t(\cdot)$ is not higher than player i's pure minimax value in the normal form game G, regardless the asynchronous move structure. This is true because the other players may always choose the pure action profile to minimax player i in every period. On the other hand, $m_i^t(\cdot)$ is not lower than player i's pure maximin value in normal form game G since it is always feasible for player i to choose the action to achieve his pure maximin value whenever player i moves. By doing so, player i guarantees himself at least his pure maximin value in every period. We now summarize these results as

Theorem 2 *Under the NEU condition,* $\forall\ i \in I$, $t \geq 0$, $I_t \in I$, *and* $a_{-I_t} \in A_{-I_t}$,

$$\max_{a_i \in A_i} \min_{a_{-i} \in A_{-i}} u_i(a) \ \leq\ m_i^t(a_{-I_t}) \ \leq\ \min_{a_{-i} \in A_{-i}} \max_{a_i \in A_i} u_i(a_i, a_{-i}).$$

The value of $m_i^t(\cdot)$, however, may be higher or lower than player i's *mixed* minimax value m_i^s. Recall the repeated BOS game where player 1 moves in every period but player 2 moves every other period. Player 2's effective minimax value cannot be more than $1/2$ in the limit as δ goes to 1, which is less than player 2's mixed minimax value $2/3$ in this case. Player 1's effective minimax value cannot be less than $5/6$ in the limit as δ goes to 1, which is more than player 1's mixed minimax value $2/3$. Whether $m_i^t(\cdot)$ is higher than m_i^s depends on the asynchronous move structure $\mathcal{I}$. When the NEU condition fails, there is no obvious relationship between $m_i^t(\cdot)$ and player i's effective minimax value (pure or mixed) in the normal form stage G.

Although $m_i^t(a_{-I_t})$ generally depends on t and a_{-I_t} for $\delta \in (0, 1)$, as δ goes to 1, the limit of $m_i^t(\cdot)$ is independent of t and a_{-I_t} under the FPI (Finite Periods Inaction) condition. The FPI condition (Yoon, 2001) requires that any player revises his action at least once during a finite number of consecutive periods. In other words,

Definition 2 *$\mathcal{I} = \{I_t\}_{t=1}^{\infty}$ satisfies the FPI (Finite Period Inaction) condition if $\exists\, T < \infty$ such that $i \in \cup_{s=t}^{t+T} I_s$ for all $t \geq 0$ and $i \in I$.*

The two examples considered in the previous section satisfy the FPI condition. Next we show that although $m_i^t(a_{-I_t})$ depends on t and a_{-I_t} (see (9)—(12)), its limit does not depend on t and a_{-I_t} as δ goes to 1. The intuition behind this result is that, it is always feasible to reach player i's worst possible initial state (to be defined) in a finite number of periods under the FPI condition. As δ goes to 1, player i's payoffs during the finite transition periods becomes negligible. What matters is the limit of player i's effective minimax value from this worst possible initial state in the future.

Theorem 3 *Under the FPI condition, $\lim_{\delta \to 1} m_i^t(a_{-I_t})$ is independent of t and a_{-I_t}.*

Proof: Under the FPI condition, $\exists\, T < \infty$ such that every player revises his action at least once between period t and period $t+T$ for any $t \geq 0$. Therefore, any initial state in period $t+T$ is reachable from any initial state in period t. Let $\hat{a}_{-I_{t+T}}$ be player i's worst initial state in period $t+T$ in the sense that

$$\lim_{\delta \to 1} m_i^{t+T}(\hat{a}_{-I_{t+T}}) = \min_{a_{-(I_{t+T} \cup E_i)}} \max_{a_{-I_{t+T} \cap E_i}} \lim_{\delta \to 1} m_i^{t+T}(a_{-I_{t+T}}) \equiv m_i. \tag{15}$$

First we show that the limit of $m_i^t(a_{-I_t})$ is not less than m_i from any initial state a_{-I_t} in period t. For any $\delta < 1$, solutions to (15) leads to some initial states in period $t+T$ which could be even worse than $\hat{a}_{-I_{t+T}}$ for player i. As $\delta \to 1$, however, the players in E_i may revise their actions optimally between periods t and $t+T$. Recursively applying (15) from period t to period $t+T$ yields

$$\begin{aligned} m_i^t(a_{-I_t}) = \max_{j \in E_i \cap I_t} \max_{a_j \in A_j} \Big\{ & (1-\delta) u_i(\hat{\sigma}_{I_t}^t(a_{-I_t}), a_{-I_t}) + \\ & \delta \max_{j \in E_i \cap I_{t+1}} \max_{a_j \in A_j} E_{\hat{\sigma}_{I_t}^t} \Big[(1-\delta) u_i(\hat{\sigma}_{I_{t+1}}^{t+1}(\tilde{a}_{-I_{t+1}}),\ \tilde{a}_{-I_{t+1}}) + \cdots + \delta E_{\hat{\sigma}_{I_{t+T}}^{t+T}} m_i^{t+T}(\cdot) \Big] \Big\} \end{aligned} \tag{16}$$

where $\tilde{a}_{-I_s}$ represents an initial state in period s induced from the initial state a_{-I_t} and the solutions to (14) $\hat{\sigma}_{I_{s'}}^{s'}$ for $t \leq s' < s \leq t+T$. Note that the first T terms on the right hand side of (16) converge to 0 as δ goes to 1 because of the common factor $(1-\delta)$. The last term on the right hand side of (16) is not less than m_i in the limit as δ goes to one. Conditions (15) and (16) imply that $\lim_{\delta\to 1} m_i^t(\cdot) \geq m_i$.

Next we show that $\lim_{\delta\to 1} m_i^t(a_{-I_t}) \leq m_i^t$. Notice that $\hat{a}_{-I_{t+T}}$ is reachable from any initial state in period t since every player revises his action at least once between period t and period $t+T$. During these T periods, the players in $-I_{t+T}$ may choose their corresponding actions in $\hat{a}_{-I_{t+T}}$ while the other players choose anything action profile which is also denoted as $\hat{a}_{I_{t+T}}$. However, action profile $\hat{a} = (\hat{a}_{-I_{t+T}}, \hat{a}_{I_{t+T}})$ may not minimize player i's lowest continuation payoff under the best unilateral deviation by those in E_i. According to (14),

$$m_i^t(a_{-I_t}) \leq \max_{j\in E_i\cap I_t} \max_{a_j\in A_j} \Bigg[(1-\delta)u_i(\hat{a}_{I_t}, a_{-I_t}) +$$
$$\delta \max_{j\in E_i\cap I_{t+1}} \max_{a_j\in A_j} \Big[(1-\delta)u_i(\hat{a}_{I_{t+1}\cup I_t}, a_{-I_{t+1}\cup I_t})) + \cdots + \delta m_i^{t+T}(\hat{a}_{-I_{t+T}}) \Big] \Bigg],$$

which implies that $\lim_{\delta\to 1} m_i^t(a_{-I_t}) \leq m_i^t$ by (15). **Q.E.D.**

Theorem 3 simplifies considerably our Folk Theorem (Theorem 4) for $G^\infty(\delta, \mathcal{I})$. Under the FPI condition, a player's effective minimax value converges to the same value in any period with any initial state. Since our Folk Theorem deals with the limiting set of subgame perfect equilibrium payoffs, what really matter are the limits of a players' effective minimax values, which depends on the stage game G and the structure of asynchronous moves $\mathcal{I}$, but not on period t and the initial state a_{-I_t} in period t.

Let $m = (m_1, \cdots, m_n)$ denote the *effective minimax point* in $G^\infty(\delta, \mathcal{I})$. By convention, any payoff vector that strictly (weakly) dominates m is strictly (weakly) *EUC* (Equivalent Utility Class) rational. Theorem 1 asserts that as δ goes to 1, any subgame perfect equilibrium payoff vector must be, at least, weakly EUC rational.

Another issue in the Folk Theorem is the feasibility. A feasible payoff vector in F may be unattainable in repeated game with asynchronous moves $G^\infty(\delta, \mathcal{I})$. For example, the feasible payoff vector (1.5, 1.5) is unattainable in the repeated BOS game with alternating moves, since switching form (2, 1) to (1, 2) involves (0, 0) under the alternating move structure. Nevertheless, Dutta (1995) argues that any feasible payoff vector can be approximated arbitrarily closely when players become sufficiently patient. We will use this result to establish our main theorem in the next section.

5 Folk Theorem

In this section, we establish the main result of this paper, an unconventional Folk Theorem for $G^\infty(\delta, \mathcal{I})$. Our Folk Theorem asserts that any feasible and strictly EUC rational payoff vector can be approximated by subgame perfect equilibrium when the players become sufficiently patient. Given Theorem 1, Theorem 4 below characterizes almost all subgame perfect equilibrium payoffs in $G^\infty(\delta, \mathcal{I})$ as δ goes to 1. Payoff vectors that are not addressed by Theorems 1 and 4 are weakly but not strictly EUC rational.

Theorem 4 *(Folk Theorem) Suppose that $G^\infty(\delta, \mathcal{I})$ satisfies the FPI condition. For any feasible and EUC rational payoff vector v ($v \in F$ and $v >> m$), $\forall\ \varepsilon > 0$, $\exists\ \underline{\delta} \in (0, 1)$ such that for all $\delta > \underline{\delta}$, $G^\infty(\delta, \mathcal{I})$ has a subgame perfect equilibrium whose average payoff vector is within ε of v.*

Proof: For any feasible and strictly EUC rational payoff vector, we first provide a strategy profile, which leads to an average payoff vector that is arbitrarily close to the target payoff vector. We then derive a set of sufficient conditions under which the strategy profile constitutes a subgame perfect equilibrium in the repeated game. Lastly, we establish all the sufficient conditions when the discount factor is large enough.

For any $v \in F$ such that $v >> m$, $\exists\ n$ personalized punishment vectors $\{v^1, v^2, \ldots, v^n\}$,

(i) $\forall\ i \in I$, vector v^i is feasible and strictly EUC rational, i.e., $v^i \in F$ and $v^i >> m$,

(ii) $\forall\ i \in I$, player i strictly prefers v to v^i, i.e., $v_i > v^i_i$,

(iii) $\forall\, i \in I$ and $j \notin E_i$, player i strictly prefers v^j to v^i, i.e., $v_i^j > v_i^i$.

Players who have equivalent utilities share the same personalized punishment vector. Notice that payoff vector v and its n personalized punishment vectors $(v^1, \cdots, v^n)$ are defined in terms of players' payoffs, which is irrelevant to whether players move synchronously or asynchronously in the repeated game. $\forall\, \varepsilon > 0$, define

$$\varepsilon^* = \min\left\{ \varepsilon,\ \min_{i \in I}\left(\frac{v_i^i - m_i}{3}\right),\ \min_{i \in I}\left(\frac{v_i - v_i^i}{3}\right),\ \min_{i \in I, j \notin E_i}\left(\frac{v_i^j - v_i^i}{3}\right)\right\} > 0. \tag{17}$$

Let $\{\pi,\ \pi^1, \cdots, \pi^n\}$ be $n+1$ outcome paths that approximate $\{v, v^1, \cdots, v^n\}$ as $\delta \to 1$, respectively. For $\varepsilon^* > 0$ by (17), $\exists\, \underline{\delta}_1 \in (0,1)$ such that for all $\delta \geq \underline{\delta}_1$, players' continuation payoffs from π in any period is within ε^* of v, and players' continuation payoffs from π^i in any period is within ε^* of v^i for all $i \in I$.

Let $\rho^i(\cdot)$ denote a path that solves $m_i^t(\cdot)$. It depends on period t and the initial state in period t. From the definition, player i's payoff from $\rho^i(\cdot)$ is less than $m_i^t(\cdot)$ in general, but is equal to $m_i^t(\cdot)$ under the best unilateral deviation by those in E_i. Theorem 3 implies that $\exists\, \underline{\delta}_2 \in (0,1)$ such that $m_i^t(\cdot)$ is within ε^* of m_i for all $\delta > \underline{\delta}_2$ for any t with any initial state. Define $M = \max_{i, a \in A} |u_i(a)| < \infty$, which is a bound of what any player receives in G.

Similar to a *simple strategy profile* of Abreu (1988), consider a strategy profile that is defined by $n+1$ outcome paths, an target path π and n punishment paths for n players. Player i's punishment path at t contains three phases. The first phase is the *effective minimax phase* that follows $\rho^i(\cdot)$ for S periods (which is to be determined). The second phase is the *transition phase* with T periods, where T is finite under the FPI condition. The third phase is the *settling phase* that starts path π^i from period $S+T$. We choose S large enough so that player i's payoff from such a punishment path is no more than

$$(1 - \delta^S)(m_i + \varepsilon^*) + (\delta^S - \delta^{S+T})M + \delta^{S+T}(v_i^i + \varepsilon^*). \tag{18}$$

Consider the following strategy profile: The strategy profile begins with the target path π. During $G^\infty(\delta, \mathcal{I})$, if player i unilaterally deviates from an ongoing path (either the target path or any of the n punishment paths), the strategy profile switches to player i's punishment path for the corresponding period and the corresponding initial state. This means that if a player deviates during his own punishment path then the strategy profile will restart player i's punishment path in the following period.

Now we derive a set of sufficient conditions under which the strategy profile described above constitutes a subgame perfect equilibrium in $G^{\infty}(\delta, \mathcal{I})$. First, consider $\delta \geq \max\{\underline{\delta}_1, \underline{\delta}_2\}$ so that our construction of the strategy profile is valid. During playing the target path π, if player $i \in I_t$ deviates in period t then player i will not receive more than M in period t. The strategy profile then calls for player i's punishment path in period $t+1$, after which player i's payoff will not be more than (18). On the other hand, if player i does not deviate then his continuation payoff will not be less than $v_i - \varepsilon^*$. Player i will not deviate during the target path π if

$$(1-\delta)M + \delta\left[(1-\delta^S)(m_i+\varepsilon^*) + (\delta^S - \delta^{S+T})M + \delta^{S+T}(v_i^i + \varepsilon)\right] \leq v_i - \varepsilon^*. \tag{19}$$

During player j's punishment path for $j \notin E_i$, player i's unilateral deviation triggers player i's punishment path. If player i deviates, his continuation payoff will not be higher than

$$(1-\delta)M + \delta\left[(1-\delta^S)(m_i+\varepsilon^*) + (\delta^S - \delta^{S+T})M + \delta^{S+T}(v_i^i + \varepsilon^*)\right] \tag{20}$$

On the other hand, if player i does not deviate then his payoff will be at least

$$(1-\delta^{S+T})(-M) + \delta^{S+T}(v_i^j - \varepsilon^*). \tag{21}$$

Player i will not deviate during player j's punishment path ($j \notin E_i$) if (20) $\leq$ (21).

During player j's punishment path for $j \in E_i$ (which is the same as the punishment path for player i), player i's deviation effectively restarts his punishment path. During the first phase, player i's deviation extends the first phase for more one period. Since player i has higher payoffs from the second and third phases than from the first phase, player i

does not benefit from any deviation during the first phase. If player i deviates during the second or the third phase then his payoff will not be more than $(1-\delta)M+\delta(m_i+\varepsilon^*)$. During the last two phases, player i receives at least $-M$ for T periods, followed by $v_i^i-\varepsilon^*$ forever. Therefore, player i will not deviate during the second and third phases in his punishment path if

$$(1-\delta)M+\delta(m_i+\varepsilon^*) \leq (1-\delta^T)(-M)+\delta^T(v_i^i-\varepsilon^*). \tag{22}$$

Recall the definition of ε^* by (17). As $\delta \to 1$, the left hand side of (19) converges to $v_i^i+\varepsilon^*$, which is strictly less than $v_i+\varepsilon^*$. For $j \notin E_i$, (20) converges to $v_i^i+\varepsilon^*$, which is strictly less than $v_i^j-\varepsilon^*$, the limit of (21) as δ goes to 1. Similarly, (22) holds with a strict inequality in the limit as $\delta \to 1$ because $m_i+\varepsilon^* < v_i^i-\varepsilon^*$. Therefore, $\exists\ \underline{\delta} > \max\{\underline{\delta}_1, \underline{\delta}_2\}$ such that inequalities (19) and (22) hold, and (20) $\leq$ (21) for all $\delta \geq \underline{\delta}$. In other words, all the sufficient conditions for the strategy profile to be a subgame perfect equilibrium are satisfied for all $\delta \in (\underline{\delta}, 1)$, which concludes this proof. **Q.E.D.**

6 Concluding Remarks

The model studied in this paper provides an integrated framework to analyze subgame perfect equilibrium in a repeated game with either synchronous moves or asynchronous moves. The conventional Folk Theorem may not fully characterize the limiting set of equilibrium payoffs when players move asynchronously. Based on the dynamic effect of players' actions on the continuation game and the effective minimax value in normal-form games, we derive a player's effective minimax value, which is determined by the stage game and the structure of players' asynchronous moves. We obtain an unconventional Folk Theorem using players' effective minimax values in the repeated games.

The model studied in this paper is closely related to the asynchronously repeated game model of Yoon (2001), where it is assumed that the players who cannot revise their actions continue to play their past actions, including past mixed actions. In this paper, we specify that the players who do not move in a period play the realizations of their past (mixed) actions. If we do not distinguish mixed and pure actions then Yoon's assumption will be equivalent to ours. This is to say, if we replace A_i with Σ_i and modify the definition of a player's effective minimax value accordingly then our unconventional Folk Theorem will characterize the limiting set of subgame perfect equilibrium payoffs in an asynchronously repeated game. Under this treatment, our Theorem 2 asserts that any player's effective minimax value is bounded between player i's mixed maximin value and mixed minimax value in the stage game. For two-player games however, the Minimax Theorem (see Fudenberg and Tirole, 1991) states that any player has the same mixed minimax and mixed maximin values. Then our Theorem 2 implies that a player's effective minimax is identical his mixed minimax in the stage game. Therefore, under the NEU condition, the conventional Folk Theorem indeed characterizes almost all perfect equilibrium payoffs in a two-player asynchronously repeated game.

As demonstrated by Takahashi and Wen (2003) however, it is not the case in games with more than two players since a player may have a higher mixed minimax value than a mixed maximin value. Because a player's effective minimax value in $G^\infty(\delta, \mathcal{I})$ is bounded from above by his mixed minimax value in G, the set of feasible and individually rational payoffs is typically a subset of what can be supported by subgame perfect equilibrium in $G^\infty(\delta, \mathcal{I})$. The limiting set of equilibrium payoffs is bounded from below by players' effective minimax values in the repeated game, rather than their minimax values in the stage game. In fact, players' effective minimax values in $G^\infty(\delta, \mathcal{I})$ depend on not only game G, but also the structure of players' asynchronous moves $\mathcal{I}$.

REFERENCES

Abreu, D. (1988): "On the Theory of Infinitely Repeated Games with Discounting," *Econometrica,* 56, 383-396.

Abreu, D., P. K. Dutta, and L. Smith (1994): "The Folk Theorem for Repeated Games: A NEU Condition," *Econometrica,* 62, 939-948.

Aumann, R. (1981): "Survey of Repeated Games," in *Essays in Game Theory and Mathematical Economics in Honor of Oskar Morgenstern*, Mannheim: Bibliographisches Institut.

Benoit, J-P., and V. Krishna (1998): "The Folk Theorems for Repeated Games: A Synthesis," Mimeo, Pennsylvania State University.

Dutta, P. K. (1995): "A Folk Theorem for Stochastic Games," *Journal of Economic Theory*, 66, 1-32.

Fudenberg, D., and E. Maskin (1986): "The Folk Theorem in Repeated Games with Discounting or with Incomplete Information," *Econometrica,* 54, 533-554.

Fudenberg, D., and J. Tirole (1991): *Game Theory.* Cambridge: The MIT Press.

Friedman, J. W. (1971): "A non-cooperative equilibrium of supergames," *Review of Economic Studies*, 38, 1-12.

Haller, H., and R. Lagunoff (2000a): "Genericity and Markovian Behavior in Stochastic Games," *Econometrica,* 68, 1231-1248.

Haller, H., and R. Lagunoff (2000b): "Markov Perfect Equilibria in Repeated Asynchronous Choice Games," Mimeo, Georgetown University.

Lagunoff, R., and A. Matsui (1997): "Asynchronous Choice in Repeated Coordination Games," *Econometrica,* 65, 1467-1477.

Lagunoff, R., and A. Matsui (2001): "Are 'Anti-Folk Theorems' in Repeated Games Nongeneric?" *Review of Economic Design,* 6, 397-412.

Rubinstein, A. (1982): "Perfect Equilibrium in a Bargaining Model," *Econometrica,* 50, 97-110.

Rubinstein, A., and A. Wolinsky (1995): "Remarks on Infinitely Repeated Extensive-Form Games," *Games and Economic Behavior,* 9, 110-115.

Shaked, A., and J. Sutton (1984): "Involuntary Unemployment as a Perfect Equilibrium in a Bargaining Model," *Econometrica,* 52, 1351-1364.

Sorin, S. (1995): "A Note on Repeated Extensive Games," *Games and Economic Behavior,* 9, 116-123.

Takahashi, S. (2003): "Effective Minimax Values in Extensive Form Games with Almost Perfect Information," Mimeo, Harvard University.

Takahashi, S., and Q. Wen (2003): "On Asynchronously Repeated Games," *Economics Letters,* 79, 239-245.

Wen, Q. (1994): "The 'Folk' Theorem for Repeated Games with Complete Information," *Econometrica,* 62, 949-954.

Wen, Q. (2002): "A Folk Theorem for Repeated Sequential Games," *Review of Economic Studies*, 69, 493-512.

Yoon, K. (2001): "A Folk Theorem for Asynchronously Repeated Games," *Econometrica,* 69, 191-200.

Yoon, K. (2004): "The Effective Minimax Value of Asynchronously Repeated Games," International Journal of Game Theory, forthcoming.