

# Two-Tier Voting: Solving the Inverse Power Problem and Measuring Inequality\*

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## Abstract

There are many situations in which different groups make collective decisions by committee voting, with each group represented by a single person. Theoretical concepts suggest how voting systems for such committees should be designed. These abstract rules can usually not be implemented perfectly, however. In this paper I address two very closely related problems. The first one is called the inverse power problem: the problem of finding voting systems that approximate the theoretical rules as best as possible. The second problem is how to measure the unequalness of a voting system. I propose a new method to address both problems, based on the coefficient of variation, and show why it should be preferred to other methods.

**Keywords:** inverse power problem; indirect voting power; measuring inequality; committee voting; assembly of representatives

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# 1 Introduction

Two-tier voting refers to situations where different groups have to make a collective decision and do so by voting in an assembly of representatives with one representative per group. Many decisions are taken daily through such voting by all kinds of institutions. The best-studied case of such two-tier voting is the Council of the European Union,<sup>1</sup> but it is by far not the only institution making use of some sort of two-tier voting. Other institutions are for example the UN General Assembly, WTO, OPEC, African Union, German Bundesrat, ECB, and thousands of boards of directors and professional and non-professional associations. While the fiercest debates so far have arisen during the reform of voting rules in the EU Council, the importance of two-tier voting is likely to increase further in the future. Globalization and the emergence of democracy in many parts of the world make collaboration in supra-national organizations more necessary and easier; furthermore, modern communication technologies facilitate the organization in interest-groups, clubs, and associations, even when the members are geographically dispersed.

The question of how such two-tier voting systems should be designed is unsolved and can in full generality certainly not be solved. Nevertheless, there are theoretical concepts that provide guidelines. These rules are often abstract and prescribe a certain distribution of power (or outcomes) rather than giving a concrete ‘operating manual’ of how the voting systems should be designed. These abstract rules can usually not be perfectly implemented. The problem of finding voting systems that approximate these theoretical rules is called the inverse power problem (‘inverse’ refers to mapping from a distribution of power to a voting system in contrast to mapping from a voting system to a distribution of power). As a first step to solve the inverse power problem, a measure is needed stating how well a voting system corresponds to a theoretical rule. It is this measure that I am concerned with in this paper.<sup>2</sup>

Another task that can be important is to measure how (un)equal voting systems are with respect to a certain variable (such a variable may for example be the probability of an individual citizen influencing the overall outcome of the two-tier voting proce-

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<sup>1</sup>The literature on two-tier voting within the EU includes, among many others, Baldwin and Widgrén (2004), Beisbart et al. (2005), Felsenthal and Machover (2004), Laruelle and Valenciano (2002), Le Breton et al. (2012), Napel and Widgrén (2006), and Sutter (2000). For an overview of promising (voting) power research avenues see Kurz et al. (2015).

<sup>2</sup>I am in particular not going to develop any algorithms addressing the inverse power problem computationally given a theoretical rule that shall be approximated and *given such a measure*, which is what most of the literature does. Finding concrete solutions to the inverse power problem is not trivial; see for example Alon and Edelman (2010), De et al. (2012), Fatima et al. (2008), Kurz (2012), Kurz and Napel (2014), Leech (2003), and De Nijs and Wilmer (2012).

ture). This measure can be used to compare voting systems within or across different constituencies. Having such a measure would for example make it possible to investigate to what degree inequality of voting systems correlates with other variables, such as income or crime rates. Furthermore, in some cases a voting system that is less equal than another one may have some advantages over the more equal one (for example it could be easier to explain its rules to citizens or this voting system could be more easily accepted by the people governed by it). In such a case it would be important to have a measure that quantifies how much more unequal one voting system is than another one rather than only having a ranking stating which voting system is less equal. As a lot of the theoretical rules on how to design voting systems in two-tier voting situations are based on equality of indirect voting power, the inverse power problem and the problem of measuring the inequality of voting systems are very closely related.

I propose to address the inverse power problem and to measure the extent of inequality of voting systems with a method based on the statistical coefficient of variation. To the best of my knowledge, I am the first to propose a measure of inequality of voting systems that can be used across different constituencies. I explain how to use this method and illustrate how it improves over other methods to address the inverse power problem. I derive and illustrate the new method alongside other methods assuming that equal indirect Banzhaf voting power is desired. However, the new method can also be applied in many other settings, including settings in which equal indirect Shapley-Shubik power is desired or settings in which the probability of success is the quantity of interest rather than voting power (see Laruelle and Valenciano, 2010).

This paper is organized as follows. In Section 2, I describe one of the possible rules prescribing how voting systems should be designed (Penrose's Square Root Rule). In Section 3, the main part of this paper, I describe the methods used so far to solve the inverse power problem and the new method based on the coefficient of variation. I also describe how the new method can be used to measure the inequality of voting systems. Furthermore, I discuss the main advantages of the new method and illustrate the differences with examples. Section 4 concludes.

## **2 One Theoretical Concept: Penrose's Square Root Rule**

In this section, I introduce one theoretical, abstract rule how voting systems should be designed, called Penrose's Square Root Rule. I will use this rule as example in the next section when talking about the inverse power problem and measuring inequality

of voting systems.<sup>3</sup> There are  $N$  different groups, numbered from 1 to  $N$ , each group  $i$  consists of  $n_i$  individuals, numbered from 1 to  $n_i$ .

**Penrose's Square Root Rule.** *The voting power of (the representative of) a group as measured by the Banzhaf index should be proportional to the square root of its population size.*

The main idea of this rule is to make it equally likely for each individual to influence the overall outcome of the two-tier voting procedure, independently of the group she belongs to. The standard motivation of this rule derives from a particular setting, which is described briefly below.<sup>4</sup>

If a winning coalition turns into a losing coalition without voter  $j$  we say that voter  $j$  has a swing. The absolute Banzhaf index of a voter  $j$  is defined as the number of possible winning coalitions that turn into losing coalitions without voter  $j$ , divided by the total number of possible coalitions. The normalized or relative Banzhaf index is the absolute Banzhaf index normalized so that the sum of the indices of all voters equals one.

Voting is binary, i.e. a proposal can either be accepted or rejected. Each individual favors the adoption of a proposal with probability one half, independently of all other individuals.<sup>5</sup> Majority voting takes place within each group and the outcome determines the vote of the representative. The representatives of all groups come together in an assembly and it is determined according to their votes and the voting system in the assembly of representatives whether a proposal is adopted or rejected.

Denote by  $\Psi_i^B$  the absolute Banzhaf power index of an individual in group  $i$  arising from majority voting in this group and by  $\Phi_i^B$  the absolute Banzhaf power index of group  $i$  in the assembly of representatives, depending on the voting system in place. Then the probability that an individual in group  $i$  has a swing with respect to the overall outcome of the voting procedure (i.e. that she influences with her vote within the group the overall outcome) is  $\Psi_i^B$  times  $\Phi_i^B$ . Thus the probability of influencing the overall outcome is equal for all individuals if  $\Psi_i^B \Phi_i^B$  is equal for all individuals or equivalently if

$$\Psi_i^B \Phi_i^B = \alpha \tag{1}$$

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<sup>3</sup>I use the most prominent rule how two-tier voting systems should be designed, but using this rule as illustration does not mean that I endorse it as a normative concept. There are different possible criticisms of this rule, see for example Laruelle and Valenciano (2008). Furthermore, it has been shown that people prefer voting systems designed according to other theoretical rules (Weber, 2014).

<sup>4</sup>The derivation of this rule in the same style can be found in more detail in Turnovec (2009); see also Turnovec et al. (2008).

<sup>5</sup>In this case, the absolute Banzhaf index of a voter is the probability that this voter has a swing.

for some constant  $\alpha$  and all  $i$ . It can easily be shown that equation (1) holds for all  $i$  if the normalized Banzhaf index of each group  $i$  is equal to

$$\frac{\frac{1}{\Psi_i^B}}{\sum_{j=1}^N \frac{1}{\Psi_j^B}}.$$

The normative rule how to design voting systems described here states that the indirect voting power  $\Psi_i^B \Phi_i^B$  should be equal for all individuals independently of which group they are in, i.e. that equation (1) should hold for all  $i$ .<sup>6</sup>

### 3 Solving the Inverse Power Problem and Measuring the Inequality of Voting Systems

In this section, I describe which methods have been used in the past to address the inverse power problem and then explain how the problem can be addressed using the coefficient of variation. I use a setting where indirect Banzhaf voting power is of interest, but the method is by no means restricted to such a setting (applying the method to other settings is straight-forward). I first describe the error terms used (which one can minimize over different voting systems to solve the inverse power problem) and then lay out the complete formulas. Then, I explain how the coefficient of variation can be used to measure how unequal voting systems are (for which the other methods do not yield good results). In the end, I illustrate why the new method based on the coefficient of variation is to be preferred over the other methods.

#### 3.1 A Method Based on Aggregated Group Level Voting Power

The system of equations (1) usually does not hold exactly for any one voting system. It is thus necessary to find an approximation, i.e. to find a voting system that corresponds as closely as possible to an equal distribution of indirect voting power across all individuals. One way to do this is to take a voting system that minimizes the deviation of the normalized Banzhaf index of each group from the vector that would yield equal indirect voting power. Taking the euclidean distance as error term, this yields minimizing

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<sup>6</sup>The reason why this is usually referred to as square root rule is the following.  $\Psi_i^B$  in equation (1) can be approximated by  $\sqrt{\frac{2}{\pi n_i}}$ , thus equation (1) holds if the Banzhaf indices of the groups are proportional to the square root of population size.

a term of the form

$$err_{group,basic}(\Psi^B, \Phi^B) := \sum_{i=1}^N \left( \frac{\Phi_i^B}{\sum_{j=1}^N \Phi_j^B} - \frac{\frac{1}{\Psi_i^B}}{\sum_{j=1}^N \frac{1}{\Psi_j^B}} \right)^2. \quad (2)$$

This term has been used in the literature to address the inverse power problem.<sup>7</sup>

It is already visible that this term cannot be the correct term to be minimized: The groups have different sizes and the idea is to equalize voting power at the individual level. I will now propose a first way to fix it and will later on compare this improved version with the method based on the coefficient of variation (as I will show later on, this easy fix does not work perfectly). As the groups have different sizes, it seems natural to also weigh the error terms by group size. In order for this error term not to increase with the number of groups or the group sizes, one can divide by the total number of individuals. Furthermore, one can take the square root, such that the error term is measured in the ‘unit’ of indirect voting power rather than in its square. For a given constituency (i.e. for fixed  $N, n_1, \dots, n_N$ ), this is just a monotonic transformation of  $err_{group,basic}$  and leads to the minimization of

$$err_{group,imp}(\Psi^B, \Phi^B) := \sqrt{\frac{1}{\sum_{i=1}^N n_i} \sum_{i=1}^N n_i \left( \frac{\Phi_i^B}{\sum_{j=1}^N \Phi_j^B} - \frac{\frac{1}{\Psi_i^B}}{\sum_{j=1}^N \frac{1}{\Psi_j^B}} \right)^2}. \quad (3)$$

### 3.2 A Method Based on Normalized Indirect Voting Power

Another method partly used in the literature to find voting systems approximating the system of equations (1) is the following. Rather than deriving the power distribution at the group level that leads to equal indirect voting power at the individual level, one considers indirect voting power  $\Psi_i^B \Phi_i^B$  directly. One then normalizes this index of indirect voting power, so that it sums up to one when added up over all individuals of group, yielding a ‘normalized indirect power index’ of the form

$$\frac{\Psi_i^B \Phi_i^B}{\sum_{j=1}^N n_j \Psi_j^B \Phi_j^B}.$$

Then one chooses a voting system that minimizes the sum of the squared deviations of this index from one over the number of individuals, so that one ends up minimizing the

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<sup>7</sup>Słomczyński and Życzkowski (2006) and Turnovec (2009) are two examples. Note that the main scientific contributions of these works are not corrupted by using this suboptimal measure.

following expression:

$$err_{indirect}(\Psi^B, \Phi^B) := \sum_{i=1}^N n_i \left( \frac{\Psi_i^B \Phi_i^B}{\sum_{j=1}^N n_j \Psi_j^B \Phi_j^B} - \frac{1}{\sum_{j=1}^N n_j} \right)^2. \quad (4)$$

I will compare this method to the other methods below. However, one comment is already in place: The reason that Penrose's Square Root Rule requires the Banzhaf power of a representative to be proportional to the square root of group size (rather than proportional to group size) is precisely that the power index used is not normalized. Therefore it seems strange to simply turn to working with a normalized version of this indirect power measure when addressing the inverse power problem.

### 3.3 A Novel Method Based on the Coefficient of Variation

Now, I introduce a new method based on the coefficient of variation. The coefficient of variation is a well established statistical concept. After introducing this concept briefly, I show how it can be derived in a meaningful way in this voting power setting.

The coefficient of variation in statistics is defined as the ratio of the (population) standard deviation  $\sigma$  to the (population) mean  $\mu$ ,

$$cv = \frac{\sigma}{\mu}.$$

It is thus the inverse of the signal-to-noise ratio. The advantage of using the coefficient of variation over the standard deviation is that the standard deviation always has to be understood in the context of the mean (e.g. multiplying all data points by two leads to a higher standard deviation but to the same coefficient of variation). The coefficient of variation is independent of the unit of measurement.

If the system of equations (1) holds, all individuals have equal (indirect) voting power. Keeping in mind that the error at the individual level is what one should be interested in, one can then minimize

$$\sum_{i=1}^N \sum_{j=1}^{n_i} (\Psi_i^B \Phi_i^B - \alpha)^2 = \sum_{i=1}^N n_i (\Psi_i^B \Phi_i^B - \alpha)^2. \quad (5)$$

over different voting systems. Remember that equal indirect voting power corresponds to equation (1) holding for *any*  $\alpha$ . Thus, it is natural to give each voting system its 'best shot', i.e. to let  $\alpha$  depend on the voting system (also the  $\Phi_i^B$  depend on the voting

system):

$$\alpha = \arg \min_{\gamma} \sum_{i=1}^N n_i (\Psi_i^B \Phi_i^B - \gamma)^2.$$

It can easily be shown that then

$$\alpha = \frac{1}{\sum_{i=1}^N n_i} \sum_{i=1}^N n_i \Psi_i^B \Phi_i^B =: \overline{\Psi^B \Phi^B}. \quad (6)$$

Note that  $\overline{\Psi^B \Phi^B}$  is the mean of  $\Psi^B \Phi^B$  (taken at the individual level).

Minimizing expression (5) with  $\alpha$  as in (6) still has some shortcomings. Also here, one would like that the error term does not necessarily depend on the number of groups or the group sizes, which can again be solved by dividing by the number of individuals. Furthermore, as for  $err_{group,imp}$ , it seems desirable to measure the variation of indirect voting power in the same unit as indirect voting power rather than its square, so again, one can take the square root. Finally, it is desirable that the scale used does not change the relevant expressions, i.e. that merely multiplying the indices  $\Phi^B$  of all groups with a constant or the unit of measurement do not change the outcome. This can be achieved by dividing through  $\overline{\Psi^B \Phi^B}$ . It turns out that one then arrives at the coefficient of variation of indirect voting power, which can thus be seen as a naturally extended version of expression (5). The new method minimizes the following expression over different voting systems (or more generally, uses this expression as a measure of inequality of voting power):

$$cv(\Psi^B \Phi^B) := \frac{\sqrt{\frac{1}{\sum_{i=1}^N n_i} \sum_{i=1}^N n_i (\Psi_i^B \Phi_i^B - \overline{\Psi^B \Phi^B})^2}}{\overline{\Psi^B \Phi^B}}. \quad (7)$$

### 3.4 Formulas (Inverse Power Problem)

The inverse power problem is to find a voting system that approximates a desired power distribution as best possible. One can of course specify a set of voting systems from which one wants to select one that best approximates this distribution. Such sets could for example be all weighted voting systems, all weighted voting systems satisfying some additional conditions (e.g. no player can be a dummy player or the quota can be at most two thirds), all double majority voting systems, or all voting system with a certain number of winning coalitions.<sup>8</sup> Denoting by  $\mathbf{W}$  the set of voting systems from which

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<sup>8</sup>Minimizing expression (2) over all weighted voting systems where the voting weights are proportional to the square root of the population size ('choosing the quota') is referred to as Jagiellonian compromise, see Słomczyński and Życzkowski (2006) and Słomczyński and Życzkowski (2010).

one wants to choose, one arrives at the following formulas that select a voting system approximating equal indirect Banzhaf voting power (now explicitly writing down the dependence of  $\Phi^B$  on the voting system  $\mathcal{W}$ ) for three of the methods discussed above (I skip the most simple version minimizing  $err_{group,basic}$ , the shortcomings of which are very easy to see):

$$\begin{aligned} \mathcal{V}_{group,imp} &= \arg \min_{\mathcal{W} \in \mathbf{W}} err_{group,imp}(\Psi^B, \Phi^B(\mathcal{W})) \\ &= \arg \min_{\mathcal{W} \in \mathbf{W}} \sqrt{\frac{1}{\sum_{i=1}^N n_i} \sum_{i=1}^N n_i \left( \frac{\Phi_i^B(\mathcal{W})}{\sum_{j=1}^N \Phi_j^B(\mathcal{W})} - \frac{1}{\sum_{j=1}^N \frac{1}{\Psi_j^B}} \right)^2}, \end{aligned} \quad (8)$$

$$\begin{aligned} \mathcal{V}_{indirect} &= \arg \min_{\mathcal{W} \in \mathbf{W}} err_{indirect}(\Psi^B, \Phi^B(\mathcal{W})) \\ &= \arg \min_{\mathcal{W} \in \mathbf{W}} \sum_{i=1}^N n_i \left( \frac{\Psi_i^B \Phi_i^B(\mathcal{W})}{\sum_{j=1}^N n_j \Psi_j^B \Phi_j^B(\mathcal{W})} - \frac{1}{\sum_{j=1}^N n_j} \right)^2, \end{aligned} \quad (9)$$

$$\begin{aligned} \mathcal{V}_{cv} &= \arg \min_{\mathcal{W} \in \mathbf{W}} cv(\Psi^B, \Phi^B(\mathcal{W})) \\ &= \arg \min_{\mathcal{W} \in \mathbf{W}} \frac{\sqrt{\frac{1}{\sum_{i=1}^N n_i} \sum_{i=1}^N n_i \left( \Psi_i^B \Phi_i^B(\mathcal{W}) - \overline{\Psi^B \Phi^B(\mathcal{W})} \right)^2}}{\overline{\Psi^B \Phi^B(\mathcal{W})}}. \end{aligned} \quad (10)$$

It can be shown that for a given constituency (i.e. for the same  $N, n_1, \dots, n_N$ )  $err_{indirect}(\cdot, \cdot)$  is just a monotonic transformation of  $cv(\cdot, \cdot)$ . This means that equations (9) and (10) yield the same results. However, it can be argued that equation (10) is built upon the right motivation and thus to be preferred. One can also see the fact that these two formulas lead to the same outcome as support for the results (but maybe not the way how one got there) achieved in research using formula (9). For measuring the inequality of voting systems in different constituencies  $err_{indirect}$  cannot be used reasonably, however, which will be shown below (furthermore, except for the ordering of voting systems according to inequality, the quantities of  $err_{indirect}$  when comparing voting systems in the same constituency are meaningless).

### 3.5 Measuring the Inequality of Voting Systems

Up to now, the question at hand has been the following: There is a constituency and one wants to see which voting system approximates an equal distribution of indirect voting power best. To solve the inverse power problem, one only needs a ranking of how unequal voting systems are for a given constituency. However, sometimes one may

also want to have a consistent measure of inequality in order to be able to compare this inequality within and across constituencies. This can be the case for different reasons: One of the reasons is that it is sometimes not only the inequality of the voting systems that matters. In some settings, efficiency can also play a role or one may care about the transparency of the voting rule, how easily the voting system can be explained to people, and how well voting systems are accepted by the populace. One may also in some cases want to compare how unequal voting systems are across constituencies, e.g. in order to be able to investigate how far the inequality of voting systems correlates with certain outcome variables (as for example income, crime rates, etc.). The coefficient of variation is designed to be a measure of variation and can thus be readily used to measure the inequality of a voting system (i.e. the variation of indirect voting power). Unlike the coefficient of variation, the error terms  $err_{group,basic}$ ,  $err_{group,imp}$ , and  $err_{indirect}$  give at best suboptimal measures of the inequality of a voting system. This will become clear in the remainder of this section.

### 3.6 Illustration of the Differences between the Methods

The methods can lead to different outcomes and the differences can be non-negligible.<sup>9</sup> Using the coefficient of variation is to be preferred over the other methods. When addressing the inverse power problem, using formulas (9) or (10) yield the same results. However, using formula (10) based on the coefficient of variation has the advantage that its motivation is better and that it is more salient – it is a well-known statistical measure and policy makers or researchers not very familiar with the topic can easily be told that the voting system is chosen with the least variation in indirect voting power (without the need to talk about normalization, etc.). To measure the inequality of voting systems, only the coefficient of variation should be used.<sup>10</sup>

In the remainder of this section I will illustrate with three (hypothetical) examples that the coefficient of variation is the best way to solve the inverse power problem and/or to measure the inequality of voting systems. The first example shows differences between

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<sup>9</sup>A first application of the new method can be found in Weber (2014).

<sup>10</sup>Another thought may be to use the (population) standard deviation. This is not optimal, however. Unlike the standard deviation, the coefficient of variation is a relative measure. This means for example that the coefficient of variation judges the indirect voting powers of two equally sized groups of individuals of 0.01 and 0.02 to exhibit more variation than 0.08 and 0.09. This relative difference is the quantity that is of interest. The absolute numbers of voting power are often not very telling. Accordingly, arguments involving indirect voting power usually involve relative numbers, for example stating that a voting system is unfair, because country  $X$ ' citizens' indirect voting power is four times as large as the one of country  $Y$ ' citizens (not that indirect voting power is 0.000002 higher in one country than in another). The example in Section 3.6.2 illustrates this difference in a bit more detail.

using the method based on  $err_{group,imp}$  and  $cv$  when the means of indirect Banzhaf voting power are equal for the voting systems, but standard deviations are different. In the second example standard deviations are equal, but means are different. The first two examples are concerned with solving the inverse power problem, i.e. with comparing voting systems with the same constituencies (in these cases using the method based on  $err_{indirect}$  yields the same results as using  $cv$ ). In the third example I compare the inequality of two voting systems in different constituencies and show how this can be done with  $cv$  while  $err_{indirect}$  does not yield reasonable results ( $err_{indirect}$  has to the best of my knowledge also not been used for this task).

### 3.6.1 First Example: Mean-Preserving Spread

There are six groups, numbered from 1 to 6. Groups 1 and 2 have ten members each, the other groups have five members. This means that in the first stage (the election of the representatives) individuals have voting power  $\Psi_{1,2}^B = 0.2460938$  and  $\Psi_{3,4,5,6}^B = 0.375$ , respectively. Indirect voting power would be equal across all individuals if the voting systems were such that

$$\frac{\Phi_{1,2}^B}{\sum_{i=1}^6 \Phi_i^B} = 0.2162162 \quad \text{and} \quad \frac{\Phi_{3,4,5,6}^B}{\sum_{i=1}^6 \Phi_i^B} = 0.1418919.$$

Now we compare two (hypothetical) voting systems  $\mathcal{W}_1$  and  $\mathcal{W}_2$ . The voting systems are such that the normalized Banzhaf indices are as follows:

$$\begin{aligned} \frac{\Phi_1^B(\mathcal{W}_1)}{\sum_{i=1}^6 \Phi_i^B(\mathcal{W}_1)} &= 0.2162162 + 0.05, & \frac{\Phi_2^B(\mathcal{W}_1)}{\sum_{i=1}^6 \Phi_i^B(\mathcal{W}_1)} &= 0.2162162 - 0.05, \\ \frac{\Phi_{3,4,5,6}^B(\mathcal{W}_1)}{\sum_{i=1}^6 \Phi_i^B(\mathcal{W}_1)} &= 0.1418919, \text{ and} \\ \frac{\Phi_{1,2}^B(\mathcal{W}_2)}{\sum_{i=1}^6 \Phi_i^B(\mathcal{W}_2)} &= 0.2162162, & \frac{\Phi_{3,4}^B(\mathcal{W}_2)}{\sum_{i=1}^6 \Phi_i^B(\mathcal{W}_2)} &= 0.1418919 + 0.05, \\ \frac{\Phi_{5,6}^B(\mathcal{W}_2)}{\sum_{i=1}^6 \Phi_i^B(\mathcal{W}_2)} &= 0.1418919 - 0.05. \end{aligned}$$

Assume for simplicity and to have a nice illustration that normalized and absolute Banzhaf indices are equal. Now we can calculate the indirect voting power of each

individual, depending on the group she is in. This yields

$$\begin{aligned}\Psi_1^B \Phi_1^B(\mathcal{W}_1) &= 0.06551414, & \Psi_2^B \Phi_2^B(\mathcal{W}_1) &= 0.04090477, \\ \Psi_{3,4,5,6}^B \Phi_{3,4,5,6}^B(\mathcal{W}_1) &= 0.05320946, \text{ and} \\ \Psi_{1,2}^B \Phi_{1,2}^B(\mathcal{W}_2) &= 0.05320946, & \Psi_{3,4}^B \Phi_{3,4}^B(\mathcal{W}_2) &= 0.07195946, \\ \Psi_{5,6}^B \Phi_{5,6}^B(\mathcal{W}_2) &= 0.03445946.\end{aligned}$$

One can easily see that using the method based on  $err_{group,imp}$  (formula (8)) does not distinguish between the methods, both voting systems would be judged to be ‘equally equal’ (the error term  $err_{group,imp}$  equals  $\sqrt{1/120}$  for both voting systems). If one looks carefully at the indirect voting power, this does not seem justified, though. For both voting systems, there are twenty individuals with indirect voting power 0.05320946, which is also the mean of indirect voting power under both voting systems. For both voting systems, there are ten individuals with higher voting power and 10 with lower power. The absolute difference between the higher value of voting power and the middle value is always equal to the difference between the middle value and the lower value; just that these differences are higher under the second voting system than under the first one. It is  $cv(\Psi^B \Phi^B(\mathcal{W}_1)) = 0.1635184$  and  $cv(\Psi^B \Phi^B(\mathcal{W}_2)) = 0.249171$ . The method based on the coefficient of variation, i.e. formula (10), thus correctly selects the first voting system,  $\mathcal{V}_{cv} = \mathcal{W}_1$ .<sup>11</sup>

### 3.6.2 Second Example: Shift of the Distribution

There are four groups. The first group has nine members while the other groups have three members each. The Banzhaf power indices in the first stage (when electing the representative) are  $\Psi_1^B = 0.2734375$  and  $\Psi_{2,3,4}^B = 0.5$ , respectively. Indirect voting power is equal for all individuals if the normalized Banzhaf indices in the assembly of representatives are

$$\frac{\Phi_1^B}{\sum_{i=1}^4 \Phi_i^B} = 0.3786982 \quad \text{and} \quad \frac{\Phi_{2,3,4}^B}{\sum_{i=1}^4 \Phi_i^B} = 0.2071006.$$

Now I compare again two voting systems,  $\mathcal{W}_1$  and  $\mathcal{W}_2$ . Assume again for simplicity that absolute and normalized Banzhaf indices are equal and assume that the two voting

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<sup>11</sup>In this example, the means of indirect voting power are equal for both voting systems. Considering the standard deviation would thus lead to the same outcome as considering the coefficient of variation. In the next example, the standard deviation is equal, while the coefficient of variation is different (see Footnote 10).

systems are such that

$$\frac{\Phi_1^B(\mathcal{W}_1)}{\sum_{i=1}^4 \Phi_i^B(\mathcal{W}_1)} = 0.3786982 - 0.09, \quad \frac{\Phi_{2,3,4}^B(\mathcal{W}_1)}{\sum_{i=1,2,3,4}^4 \Phi_i^B(\mathcal{W}_1)} = 0.2071006 + 0.03,$$

and

$$\frac{\Phi_1^B(\mathcal{W}_2)}{\sum_{i=1}^4 \Phi_i^B(\mathcal{W}_2)} = 0.3786982 + 0.09, \quad \frac{\Phi_{2,3,4}^B(\mathcal{W}_2)}{\sum_{i=1,2,3,4}^4 \Phi_i^B(\mathcal{W}_2)} = 0.2071006 - 0.03.$$

Which of these two voting systems do the methods select? For the method based on  $err_{group,imp}$  the two systems approximate equal indirect voting power equally well. This can be seen as follows. The terms in parentheses in expression (8) for both voting systems are always either  $-0.09$  or  $+0.09$  for the parts referring to the large group and either  $+0.03$  or  $-0.03$  for the parts referring to the small groups. As only the squares of these values enter expression (8), these two voting systems are ‘equally equal’ when judged by this method. The method based on the coefficient of variation, in contrast, does make a difference between these two voting systems. The indirect voting power of each individual is under the first voting system

$$\Psi_1^B \Phi_1^B(\mathcal{W}_1) = 0.07894092 \quad \text{and} \quad \Psi_{2,3,4}^B \Phi_{2,3,4}^B(\mathcal{W}_1) = 0.1185503,$$

and under the second voting system

$$\Psi_1^B \Phi_1^B(\mathcal{W}_2) = 0.1281597 \quad \text{and} \quad \Psi_{2,3,4}^B \Phi_{2,3,4}^B(\mathcal{W}_2) = 0.0885503.$$

Remember that exactly half of the individuals are in the large group. While under the first voting system an individual in the half of the population with more indirect voting power holds 1.50176 times as much indirect voting power as an individual in the other half of the population, this ratio is only 1.447309 under the second voting system.<sup>12</sup> The coefficient of variation is 0.2005627 for  $\mathcal{W}_1$  and 0.182776 for  $\mathcal{W}_2$ . Thus, this method selects the second voting system,  $\mathcal{V}_{cv} = \mathcal{W}_2$ .

### 3.6.3 Third Example: Comparing the Inequality of Voting Systems across Constituencies

So far, I have assumed that one wants to compare voting systems in the same constituency (i.e.  $N, n_1, \dots, n_N$  are fixed). As explained above, in some cases one also wants

<sup>12</sup>The absolute values of the differences  $\Psi_1^B \Phi_1^B(\mathcal{W}_1) - \Psi_{2,3,4}^B \Phi_{2,3,4}^B(\mathcal{W}_1)$  and  $\Psi_1^B \Phi_1^B(\mathcal{W}_2) - \Psi_{2,3,4}^B \Phi_{2,3,4}^B(\mathcal{W}_2)$  are equal.

to measure how unequal voting systems are and compare them when their constituencies are different. This can be done meaningfully with the coefficient of variation, but not with the error term  $err_{indirect}$  (neither with the other error terms, which I do not use again in this example).

The first constituency consists of six groups of four people each. The second constituency consists of four groups of eight people each. Assume that the voting system in place in the first constituency ( $\mathcal{W}_X$ ) is such that indirect voting power is as follows:

$$\Psi_{1,2,3}^B \Phi_{1,2,3}^B(\mathcal{W}_X) = 0.03 \quad \text{and} \quad \Psi_{4,5,6}^B \Phi_{4,5,6}^B(\mathcal{W}_X) = 0.01.$$

In the second constituency, the voting system ( $\mathcal{W}_Y$ ) is such that indirect voting power is as follows:

$$\Psi_{1,2}^B \Phi_{1,2}^B(\mathcal{W}_Y) = 0.03 \quad \text{and} \quad \Psi_{3,4}^B \Phi_{3,4}^B(\mathcal{W}_Y) = 0.01.$$

This means that in the first constituency half of the individuals have indirect voting power of 0.03, while the other individuals have voting power 0.01. The same holds for the second constituency. Which of the two voting systems is more unequal? The only reasonable answer is that they are ‘equally unequal’. Using the coefficient of variation as measure of inequality also yields this result: It is  $cv(\Psi^B, \Phi^B(\mathcal{W}_X)) = 0.5 = cv(\Psi^B, \Phi^B(\mathcal{W}_Y))$ . However, if one tried to use  $err_{indirect}$  as a measure of inequality, one would obtain  $err_{indirect}(\Psi^B, \Phi^B(\mathcal{W}_X)) = 0.006944444$  and  $err_{indirect}(\Psi^B, \Phi^B(\mathcal{W}_Y)) = 0.0078125$ . One would then misleadingly conclude that the voting system in the first constituency is less unequal than the voting system in the second constituency.

## 4 Concluding Remarks

I have introduced a method to address the inverse power problem in two-tier voting settings and to measure the inequality of voting systems. This method is based on the statistical coefficient of variation and can be used in many different settings. After deriving it in a setting where equal indirect Banzhaf voting power is desired, I have shown why this method is to be preferred over other methods. The main advantage of the new method is that it yields the correct results. Further advantages of the method are that it is more intuitive and more salient for policy makers or researchers not familiar with the topic.

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