A Characterization of the Position Value

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Abstract

We characterize the position value for arbitrary communication situations. The two properties involved in the characterization are component efficiency, which is standard, and balanced total threats, which is in the same spirit as balanced contributions. Since the Myerson value can be characterized by component efficiency and balanced contributions a comparison between the two allocation rules based on characterizing properties can be made.

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1 Introduction

The study of cooperative games with restricted cooperation possibilities is well documented. The first to model restricted cooperation by means of an undirected graph was Myerson (1977). He introduced communication situations, which consist of a cooperative game and an undirected graph. The vertices in the graph correspond to the players in the cooperative game and the edges between the players correspond to bilateral communication possibilities. For these situations, Myerson (1977) introduced and characterized an allocation rule. Later on, alternative characterizations, some valid on restricted sets of communication situations only, where given in Myerson (1980) and Borm et al. (1992).

The main contribution of Borm et al. (1992) was the introduction of an alternative allocation rule for communication situations. This rule is called the position value. Borm et al. (1992) provide a characterization of this rule for communication situations with a tree as the underlying graph only. They conclude by stating the open problem how to characterize the position value for the class of all communication situations. As far as we know, no satisfactory answer to this question has been given so far.[1]

The main contribution of this paper is a characterization of the position value for arbitrary communication situations. Specifically, this implies that no specific requirements on the underlying graph are needed. The two properties involved in this characterization are component efficiency, which is standard in almost all characterizations of the position value and the Myerson value, and balanced total threats. This latest property deals with threats between players. Such a threat is based on payoff differences a player can inflict on another player by breaking one of the links this first player is involved in. The threat of a player towards another player is defined as the sum of these differences over the links of the first player, i.e., as the sum of the payoff differences a player can inflict on another player by breaking one of his links. Balanced total threats states that the threat from a player towards another player equals the reverse threat. This property is in the same spirit as balanced contributions, which in conjunction with component efficiency characterizes the Myerson value (See Myerson (1980)). This allows for a solid comparison between the values based on underlying properties.

The recent literature on communication situations, or more generally, on networks in cooperative situations increasingly concentrates on network formation. Slikker and van den Nouweland (2001) provide a recent review, both on the cooperative, axiomatic approaches to allocation rules and on the formation issues.

[1]Slikker (1999) proves that for reward communication situations there exist a unique link potential and a unique player potential, where reward communication situations can be seen as a superset of the set of communication situations. The marginal contributions of the players to these potentials correspond to the position value and the Myerson value, respectively. This is in the same spirit as the potential for the Shapley value of Hart and Mas-Colell (1989), who remark (Remark 2.5) that their potential approach can be seen as a new characterization. This would contradict the statement we just made. Though useful, e.g., for proving results for general classes of games, we feel that the potential approach does not seem to be generally accepted as a (solid) characterization. Finally, for the sake of completeness, we remark that the potential approach of Slikker (1999) can be restricted to communication situations only, thereby perhaps losing some of its original appeal.

The setup of the remainder of this paper is as follows. In Section 2 we introduce some notation and definition. This section contains the formal descriptions of the position value and the Myerson value. The main result of the paper, a characterization of the position value, can be found in Section 3. We conclude this paper in Section 4 with a discussion on the comparison of the position value and the Myerson value based on characterizing properties.

2 Preliminaries

In this section we present some notations and definitions that will be used in the remainder of this paper.

A cooperative game is a pair (N, v), where $N = \{1, \ldots, n\}$ denotes the set of players and $v : 2^N \to \mathbb{R}$ with $v(\emptyset) = 0$ the characteristic function. A game is called zero-normalized if $v(\{i\}) = 0$ for all $i \in N$. For a coalition $S \subseteq N$, $v_{|S}$ denotes the restriction of the characteristic function v to the player set S, i.e., $v_{|S}(T) = v(T)$ for each coalition $T \subseteq S$. The pair $(S, v_{|S})$ is a cooperative game with player set S that is obviously closely related to the game (N, v).

Let N be a set of players and let $R \in 2^N \backslash \{\emptyset\}$. The *unanimity game* (N, u_R) is the game with $u_R(S) = 1$ if $R \subseteq S$ and $u_R(S) = 0$ otherwise (see Shapley (1953)). Every game (N, v) can be written as a linear combination of unanimity games in a unique way, i.e., $v = \sum_{R \in 2^N \backslash \{\emptyset\}} \lambda_R(v) u_R$. The *Shapley value* Φ of a game is now easily described by[2]

$$\Phi_i(N, v) = \sum_{R \subseteq N:\ i \in R} \frac{\lambda_R(v)}{|R|} \text{ for all } i \in N.$$

A (*communication*) *graph* is a pair (N, L) where the set of vertices N represents the set of players and the set of edges $L \subseteq L^N = \{\{i, j\} \mid \{i, j\} \subseteq N,\ i \neq j\}$ represents the set of bilateral (communication) links. Two players i and j are *directly connected* iff $\{i, j\} \in L$. Two players i and j are *connected* (directly or indirectly) iff $i = j$ or there exists a path between players i and j, i.e., if there exists a sequence of players $(i_1, i_2, \ldots, i_t)$ such that $i_1 = i$, $i_t = j$, and $\{i_k, i_{k+1}\} \in L$ for all $k \in \{1, 2, \ldots, t-1\}$. The notion of connectedness induces a partition of the player set into communication components, where two players are in the same *communication component* if and only if they are connected. The set of communication components of (N, L) will be denoted by N/L. Furthermore, denote the subgraph on the vertices in coalition $S \subseteq N$ by $(S, L(S))$, where $L(S) = \{\{i, j\} \in L \mid \{i, j\} \subseteq S\}$, and the partition of S into communication components according to graph $(S, L(S))$ by S/L. Finally, for all $i \in N$ define

$$L_i = \Big\{ l \in L \,\Big|\, i \in l \Big\}, \tag{1}$$

the set of links player i is involved in.

Myerson (1977) studied communication situations (N, v, L) where (N, v) is a cooperative game and (N, L) a communication graph. He introduced the *graph-restricted game* (N, v^L), where

$$v^L(S) = \sum_{C \in S/L} v(C) \text{ for all } S \subseteq N.$$

[2] $|R|$ denotes the cardinality of a set R.

So, a coalition is split into communication components and the value of this coalition in the graph-restricted game is then defined as the sum of the values of the communication components in the original game. The Shapley value of the game (N, v^L) is usually referred to as the *Myerson value* of communication situation (N, v, L). Notation:

$$\mu(N, v, L) = \Phi(N, v^L).$$

An alternative allocation rule for communication situations, the position value, was introduced by Meessen (1988) and Borm et al. (1992). Let (N, v, L) be a communication situation. For the game (N, v) let the associated link game (L^N, r^v) be defined by

$$r^v(A) = \sum_{C \in N/A} v(C), \text{ for all } A \subseteq L^N.$$

This link game can be seen as a cooperative game in which the players can be identified with pairs of players in the original game. Hence, the link game can be written as a unique linear combination of unanimity link games, i.e.,

$$r^v = \sum_{A \subseteq L^N} \alpha_A^v u_A.$$

Note that though these games are called unanimity link games, they will in general not be the link game associated with a cooperative game. Furthermore, note that for any $L \subseteq L^N$ the restriction of (L^N, r^v) can be described similarly, i.e.,

$$r_{|L}^v = \sum_{A \subseteq L} \alpha_A^v u_A.$$

We remark that we use α_A^v rather than $\lambda_A(r^v)$ to denote the unanimity coefficient of set of links A in the link game (L^N, r^v) associated with cooperative game (N, v).

The position value attributes to each player half of the value of each link he is involved in, where the value of a link is defined as the payoff the Shapley value attributes to this link in the associated (restricted) link game $(L, r_{|L}^v)$. Formally, the *position value* π of communication situation (N, v, L) with (N, v) zero-normalized is defined by

$$\pi_i(N, v, L) = \sum_{l \in L_i} \frac{1}{2} \Phi_l(L, r_{|L}^v), \text{ for all } i \in N. \tag{2}$$

Throughout this work we restrict ourselves to communication situations with a zero-normalized underlying cooperative game.[3]

[3]For communication situations with arbitrary (N, v), the position value of a player is defined as the sum of his individual value and his position value of the communication situation with the zero-normalization of (N, v) as the underlying game. Our results hold for this more general setting, but would require additional notation, which would only distract from the main result. Therefore, we restrict to zero-normalized cooperative games.

3 The characterization

In this section we prove the main result of this paper, a characterization of the position value for communication situations. Opposed to earlier characterizations no condition on the underlying graph is required.

Consider the following properties for an allocation rule γ defined on a class of communication situations D:

Component efficiency (CE): For all $(N, v, L) \in D$ and all $C \in N/L$ it holds that

$$\sum_{i \in C} \gamma_i(N, v, L) = v(C). \tag{3}$$

Balanced total threats (BTT): For all $(N, v, L) \in D$ and all $i, j \in N$ it holds that

$$\sum_{l \in L_j} \Big[\gamma_i(N, v, L) - \gamma_i(N, v, L\backslash\{l\})\Big] = \sum_{l \in L_i} \Big[\gamma_j(N, v, L) - \gamma_j(N, v, L\backslash\{l\})\Big]. \tag{4}$$

The first property is standard and dates back to Myerson (1977). Balanced total threats deals with the loss players can inflict on each other. The total threat of a player towards another player is defined as the sum over all links of the first player of the payoff differences the second player experiences if such a link is broken. Balanced total threats states that the total threat of a player towards another player is equal to the reverse total threat.

Before we prove that the position value satisfies these two properties, we provide an alternative description of the position value. Recall from Section 2 that the link game (L^N, r^v) associated with a game (N, v) can be written as a unique linear combination of unanimity link games, i.e.,

$$r^v = \sum_{A \subseteq L^N} \alpha^v_A u_A.$$

The position value of communication situation (N, v, L) can now be described in terms of the unanimity coefficients of the associated link game (cf. Slikker (1999)), since for all $i \in N$ it holds that

$$\pi_i(N, v, L) = \sum_{l \in L_i} \frac{1}{2} \Phi_l(L, r^v_{|L}) = \sum_{l \in L_i} \frac{1}{2} \sum_{A \subseteq L:\ l \in A} \frac{\alpha^v_A}{|A|} = \sum_{A \subseteq L} \frac{1}{2} \alpha^v_A \frac{|A_i|}{|A|}, \tag{5}$$

where the second equality follows by the description of the Shapley value in Section 2. Using this expression we can easily show that the position value satisfies component efficiency and balanced total threats.

Lemma 3.1 The position value satisfies component efficiency and balanced total threats.

Proof: Component efficiency of the position value was already proven by Borm et al. (1992). It remains to prove that the position value satisfies balanced total threats. Let (N, v, L) be a communication situation and let $i, j \in N$ be two distinct players. Then

$$\begin{aligned}\sum_{l \in L_j} \Big[\pi_i(N,v,L) - \pi_i(N,v,L\backslash\{l\})\Big] &= \frac{1}{2}\sum_{l\in L_j}\Big[\sum_{A\subseteq L}\alpha_A^v \frac{|A_i|}{|A|} - \sum_{A\subseteq L\backslash\{l\}}\alpha_A^v\frac{|A_i|}{|A|}\Big] \\ &= \frac{1}{2}\sum_{l\in L_j}\sum_{A\subseteq L\colon\, l\in A}\alpha_A^v\frac{|A_i|}{|A|} \\ &= \frac{1}{2}\sum_{A\subseteq L}|A_j|\alpha_A^v\frac{|A_i|}{|A|} \\ &= \frac{1}{2}\sum_{A\subseteq L}\alpha_A^v\frac{|A_i|\cdot|A_j|}{|A|} \\ &= \sum_{l\in L_i}\Big[\pi_j(N,v,L)-\pi_j(N,v,L\backslash\{l\})\Big],\end{aligned}$$

where the first equality follows by definition, the second to fourth equality by rearranging terms and the last equality follows similar to the first four (note that the expression after the fourth equality sign is symmetric in i and j). □

We illustrate this lemma in the following example.

Example 3.1 Consider communication situation (N, v, L) with $N = \{1, 2, 3\}$, $v = u_{\{1,2\}}$, and $L = \{\{1, 2\}, \{1, 3\}, \{2, 3\}\}$. The payoffs for several (sub-)graphs according to the position value are given in table 1.

graph A	$\pi(N, v, A)$
$\{\{1,2\},\{1,3\}\}$	$(\frac{1}{2},\frac{1}{2},0)$
$\{\{1,2\},\{2,3\}\}$	$(\frac{1}{2},\frac{1}{2},0)$
$\{\{1,3\},\{2,3\}\}$	$(\frac{1}{4},\frac{1}{4},\frac{1}{2})$
L	$(\frac{5}{12},\frac{5}{12},\frac{1}{6})$

Table 1: Position values for several graphs.

Then the threat of player 1 towards player 2 equals $(\frac{5}{12} - \frac{1}{4}) + (\frac{5}{12} - \frac{1}{2}) = \frac{1}{12}$ by breaking the link with player 2 and that with player 3, respectively. Obviously, the reverse threat is the same since players 1 and 2 are symmetric.

The threat of player 1 towards player 3 equals $(\frac{1}{6} - \frac{1}{2}) + (\frac{1}{6} - 0) = -\frac{2}{12}$ by breaking the link with player 2 and that with player 3, respectively. The reverse threat of player 3 towards player 1 equals $(\frac{5}{12} - \frac{1}{2}) + (\frac{5}{12} - \frac{1}{2}) = -\frac{2}{12}$ as well.

This illustrates that according to the position value total threats are balanced indeed. ◇

We will use lemma 3.1 in the following theorem.

Theorem 3.1 The position value is the unique allocation rule for communication situations that satisfies component efficiency and balanced total threats.

Proof: By lemma 3.1 we know that the position value satisfies CE and BTT. Suppose γ satisfies CE and BTT. We will show that γ coincides with π. The proof will be by induction to $|L|$.

First note that for all (N, v, L) with $|L| = 0$ it follows directly that γ and π coincide by CE of the two allocation rules.

Secondly, let $k \geq 1$ and suppose that γ and π coincide for all (N, v, L) with $|L| \leq k-1$. Let (N, v, L) be such that $|L| = k$. We will show for all $C \in N/L$ that $\gamma_i(N, v, L) = \pi_i(N, v, L)$ for all $i \in C$. This equality follows directly from CE for all $C \in N/L$ with $|C| = 1$. Let $C \in N/L$ with $|C| \geq 2$. Without loss of generality denote $C = \{1, 2, \ldots, c\}$. By BTT and CE we have the following system of equalities.[4]

$$\begin{aligned}
|L_2|\gamma_1(L) - |L_1|\gamma_2(L) &= \sum_{l \in L_2} \gamma_1(L \backslash \{l\}) - \sum_{l \in L_1} \gamma_2(L \backslash \{l\}) \\
&\cdots \\
|L_c|\gamma_1(L) - |L_1|\gamma_c(L) &= \sum_{l \in L_c} \gamma_1(L \backslash \{l\}) - \sum_{l \in L_1} \gamma_c(L \backslash \{l\}) \\
\sum_{i \in C} \gamma_i(L) &= v(C)
\end{aligned}$$

The first $c-1$ equalities come from BTT applied to pairs $[1, j]$ for $j \in \{2, \ldots, c\}$. Note that $|L_j| \geq 1$ for all $j \in \{1, \ldots, c\}$ since $|C| \geq 2$ and $j \in C$ for all $j \in \{1, \ldots, c\}$. The last equality comes from CE.

By the induction hypothesis it follows that the system above is equivalent to

$$\begin{aligned}
|L_2|\gamma_1(L) - |L_1|\gamma_2(L) &= \sum_{l \in L_2} \pi_1(L \backslash \{l\}) - \sum_{l \in L_1} \pi_2(L \backslash \{l\}) \\
&\cdots \\
|L_c|\gamma_1(L) - |L_1|\gamma_c(L) &= \sum_{l \in L_c} \pi_1(L \backslash \{l\}) - \sum_{l \in L_1} \pi_c(L \backslash \{l\}) \\
\sum_{i \in C} \gamma_i(L) &= v(C)
\end{aligned}$$

It is a straightforward exercise to show that this is a regular system in c variables, $\gamma_1(L), \ldots, \gamma_c(L)$. Consequently, this system has a unique solution. Since the position value satisfies BTT and CE we know that $(\pi_1(L), \ldots, \pi_c(L))$ is a solution, and, hence, the unique solution.

We conclude that $\gamma(N, v, L) = \pi(N, v, L)$ for all (N, v, L) with $|L| = k$. □

[4] We suppress N and v and write, for example, $\gamma(L)$ instead of $\gamma(N, v, L)$.

4 Discussion

Theorem 3.1 provides a characterization of the position value for arbitrary communication situations. This is in line with the work of Borm et al. (1992), in which characterizations are provided in case the underlying graph is a tree. Furthermore, on the same restricted class they provide a similar characterization for the Myerson value. Both characterizations use the properties component efficiency, additivity, and the superfluous link property. Additionally, the characterization of the position value uses link anonymity, whereas in the characterization of the Myerson value point anonymity is used. On the set of communication situations with a tree as underlying graph the difference between the position value and the Myerson value can be traced to the difference between link anonymity and point anonymity.[5]

The characterization in this paper clears the way for a comparison between the position value and the Myerson value on the class of all communication situations. The characterization of the Myerson value that comes closest to the characterization in this paper is the one that can be traced back to Myerson (1980). The two properties involved are component efficiency and balanced contributions. We will define this latest property for an allocation rule γ defined on a class of communication situations D:

Balanced contributions (BC): For all $(N, v, L) \in D$ and all $i, j \in N$ it holds that

$$\gamma_i(N, v, L) - \gamma_i(N, v, L \backslash L_j) = \gamma_j(N, v, L) - \gamma_j(N, v, L \backslash L_i). \tag{6}$$

In fact, balanced contributions can be seen as a balanced threat property as well. In doing so, the difference between the position value and the Myerson value comes down to a difference in measuring threats. For the Myerson value, this threat is measured by the payoff difference a player can inflict on another by breaking all his links. For the position value, this threat is measured by the sum of the possible payoff differences a player can inflict on another by breaking one of his links.

By rewriting the threat for the Myerson value, the two properties can be compared on an even more detailed level. Let (N, v, L) be a communication situation and let $i, j \in N$. Write $L_j = \{1, \ldots, l_j\}$. Then

$$\begin{aligned}\gamma_i(N, v, L) - \gamma_i(N, v, L \backslash L_j) = & \Big(\gamma_i(N, v, L) - \gamma_i(N, v, L \backslash \{l_1\})\Big) \\ & + \Big(\gamma_i(N, v, L \backslash \{l_1\}) - \gamma_i(N, v, L \backslash \{l_1, l_2\})\Big) \\ & + \ldots + \Big(\gamma_i(N, v, L \backslash \{l_1, \ldots, l_{j-1}\}) - \gamma_i(N, v, L \backslash \{l_1, \ldots, l_j\})\Big).\end{aligned}$$

Hence, both the threat for the position value as the one for the Myerson value can be seen as the sum of the possible payoff differences a player can inflict on another player by breaking one of his links. However, when breaking these links one-by-one, for the position value a broken

[5] Borm et al. (1992) refer to superfluous link property, link anonymity, and point anonymity as superfluous arc property, degree property, and communication ability property, respectively. Our terminology is in line with Slikker and van den Nouweland (2001).

link should be restored before breaking the next one, whereas this link should not be restored for the Myerson value.

Of course, similar comparisons can be made when looking at the properties from a contribution rather than from a threat point of view. Balanced contributions measures the contribution of a player to the payoff of another player by the payoff difference for the second player compared with the situation in which all links of the first player have been removed. For balanced total threats, this contribution is regarded as the sum of the contributions of each of the links of the first player, i.e., for each of the links of the first player look at the payoff difference for the second player if this link is removed (and all other links of the first player are present) and subsequently, sum these differences. According to both properties, contributions should be balanced.

Concluding, we stress that besides the fact that the characterization of the position value is interesting in itself, it provides us with the opportunity to make a solid comparison between characterizing properties of the position value and the Myerson value on the class of all communication situations.

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