PROOFS OF APPENDIX A

Proof of PROPOSITION A1 (pure strategy Bayesian-Nash equilibria in the PU-participation game without allied voters):

The proof is a straightforward probabilistic extension of Palfrey and Rosenthal (1983). Note that in our case we assume $N_i \geq 1$, $i = A, B$.

(*i*): It is easy to see [*cf.* condition (7)] that if $c > 1/2$, participation is too costly for any voter and full abstention is the only equilibrium.

(*ii*) to (*iv*): If $c < 1/2$, in order for turnout $V = 0,1,...,E$, $V = V_i + V_{-i}$, to be a pure strategy Bayesian-Nash equilibrium outcome, no (non-)participant may receive a strictly higher expected payoff by deviating to abstention (participation).

For V *even*, every decision is pivotal when $V_i = V_{-i}$. In all other cases, nobody is pivotal because $|V_i - V_{-i}| \geq 2$. This implies that for $V_i = V_{-i}$ changing one's decision affects revenues and for $V_i \neq V_{-i}$ it does not. Using this, we can derive necessary and sufficient conditions for pure strategy Bayesian-Nash equilibria with even turnout V to exist.

First, the expected increase in revenue if a non-participant decides to vote must be equal to or smaller than the costs:

$$\Phi(V, v_{j_i} = 0)\frac{1}{2} \leq c, \qquad (A1)$$

where $\Phi(V, v_{j_i} = 0)$ is the probability that the vote will affect the outcome, which (because V is even) only occurs if there is a tie:

$$\Phi(V, v_{j_i} = 0) = prob(V_i = V_{-i} = V/2 | V, v_{j_i} = 0)$$

$$= \sum_{x=\max[\underline{N}_i, V/2+1]}^{\min[\overline{N}_i, E-V/2]} prob(x)\, prob(V_i = V_{-i} = V/2 | V, v_{j_i} = 0, x).$$

Second, the expected decrease in revenue if a participant decides to abstain must be equal to or larger than the costs saved:

$$\Phi(V, v_{j_i} = 1)\frac{1}{2} \geq c, \qquad (A2)$$

where $\Phi(V, v_{j_i} = 1)$ denotes the probability that the switch will affect the outcome, which (because V is even) only occurs if there is a tie:

$$\Phi(V, v_{j_i} = 1) = prob(V_i = V_{-i} = V/2 | V, v_{j_i} = 1)$$

$$= \sum_{x=\max[\underline{N}_i, V/2]}^{\min[\overline{N}_i, E-V/2]} prob(x)\, prob(V_i = V_{-i} = V/2 | V, v_{j_i} = 1, x).$$

For V *odd*, we first establish the cases where an abstainer is pivotal. This occurs when $V_i = V_{-i} - 1$, implying that j_i can force a tie. If, however, j_i is one of the V participants, (s)he is pivotal if $V_i = V_{-i} + 1$, because a switch to abstention would reduce a victory to a tie. In all other cases $|V_i - V_{-i}| \geq 3$ and changing one's decision has no effect on revenues. Using this, we can derive necessary and sufficient conditions for pure strategy Bayesian-Nash equilibria with (odd) turnout V to exist.

First, for a non-participant, $(\text{A}1)$ must hold. Now, $\Phi(V, v_{j_i} = 0)$ is given by the probability that j_i is in a group with a one-vote defeat to the other group:

$$\Phi(V, v_{j_i} = 0) = prob(V_i = V_{-i} - 1 = \lfloor V/2 \rfloor | V, v_{j_i} = 0)$$

$$= \sum_{x=\max[\underline{N}_i, \lceil V/2 \rceil]}^{\min[\overline{N}_i, E - \lceil V/2 \rceil]} prob(x) prob(V_i = V_{-i} - 1 = \lfloor V/2 \rfloor | V, v_{j_i} = 0, x).$$

Second, for a participant, $(\text{A}2)$ must hold. Here, $\Phi(V, v_{j_i} = 1)$ is the probability that j_i is in a group with a one-vote victory over the other group:

$$\Phi(V, v_{j_i} = 1) = prob(V_i = V_{-i} + 1 = \lceil V/2 \rceil | V, v_{j_i} = 1)$$

$$= \sum_{x=\max[\underline{N}_i, \lceil V/2 \rceil]}^{\min[\overline{N}_i, E - \lfloor V/2 \rfloor]} prob(x) prob(V_i = V_{-i} + 1 = \lceil V/2 \rceil | V, v_{j_i} = 1, x).$$

Next, we investigate whether and which pure strategy Bayesian-Nash equilibria exist that fulfill $(\text{A}1)$ and $(\text{A}2)$. We consider all possible cases $0 \leq V \leq E$.

Full abstention ($V = 0$):

Full abstention cannot be an equilibrium. We only need to consider $(\text{A}1)$, because $v_{j_i} = 0$, $\forall j_i$, $i = A, B$. This reduces to

$$prob(V_i = V_{-i} = 0 | V = 0, v_{j_i} = 0) = \sum\nolimits_{x=\underline{N}_i}^{\overline{N}_i} prob(x) \cdot 1 = 1 \leq 2c, \qquad (\text{A}3)$$

since $prob(V_i = V_{-i} = 0 | V = 0, v_{j_i} = 0, x) = 1$, $\forall x$, which contradicts our assumption that $c < 1/2$.

Full abstention in one group and positive participation in the other group ($V_i = 0; V_{-i} > 0$):

Full abstention $V_i = 0$ in i and positive participation $V_{-i} > 0$ in $-i$ cannot be an equilbrium. This is easy to see. Given the pure strategy of abstention followed by everyone in *i*, for any $V_{-i} > 1$, every participant in $-i$ has an incentive to abstain until $V_{-i} = 1$, which suffices to win the election. But if $V_{-i} = 1$, it is advantageous for every abstainer in i to participate, because the value from turnout is $1/2$, which exceeds $c < 1/2$.

Full participation ($V = E$):

For some $c < 1/2$, equilibria with full participation exist. We only need to consider (A2), because $v_{j_i} = 1,\ \forall j_i,\ i = A, B$. For E even {odd} this reduces to

$$prob(V_i = V_{-i} = E/2 \mid V = E, v_{j_i} = 1) = prob(x = E/2) \cdot 1 \geq 2c \qquad \text{(A4)}$$

$$\{\ prob(V_i = V_{-i} + 1 = \lceil E/2 \rceil \mid V = E, v_{j_i} = 1) = prob(\ x = \lceil E/2 \rceil) \cdot 1 \geq 2c\ \}.$$

Hence, for $c \leq prob(x = E/2)/2 = prob(N_i = N_{-i})/2$ $\{\ c \leq prob(x = \lceil E/2 \rceil)/2 = prob(N_i = N_{-i} + 1)/2\ \}$ (A2) is satisfied, which proves (*ii*) of the proposition.

Full participation in one group and possibly some in the other group ($V_i = x,\ V_{-i} < E - x$):

$V = V_i = x = 1$ cannot be an equilibrium because voters in $-i$ have an incentive to switch to voting. For $1 < V < E$, there exist equilibria for some $c < 1/2$ with full participation $V_i = x$ in i and possibly some participation $V_{-i} < E - x$ in $-i$. For V even {odd}, (A1) applied to $-i$ gives

$$prob(V_i = V_{-i} = x \mid V = 2x, v_{j_{-i}} = 0) = prob(x = V/2) \cdot 1 \leq 2c$$

$$\{\ prob(V_i = V_{-i} + 1 = x \mid V = 2x - 1, v_{j_{-i}} = 0) = prob(x = \lceil V/2 \rceil) \cdot 1 \leq 2c\ \}$$

and (A2) applied to i gives

$$prob(V_i = V_{-i} = x \mid V = 2x, v_{j_i} = 1) = prob(x = V/2) \cdot 1 \geq 2c$$

$$\{\ prob(V_i = V_{-i} + 1 = x \mid V = 2x - 1, v_{j_i} = 1) = prob(\ x = \lceil V/2 \rceil) \cdot 1 \geq 2c\ \}$$

and to $-i$

$$prob(V_i = V_{-i} = x \mid V = 2x, v_{j_{-i}} = 1) = prob(x = V/2) \cdot 1 \geq 2c$$

$$\{\ prob(V_i = V_{-i} - 1 = x \mid V = 2x + 1, v_{j_{-i}} = 1) = prob(\ x = \lfloor V/2 \rfloor) \cdot 1 \geq 2c\ \}. \qquad \text{(A5)}$$

Hence, $c = prob(x = V/2)/2$ $\{\ c = prob(x = \lceil V/2 \rceil)/2 \wedge c \leq prob(x = \lfloor V/2 \rfloor)/2\ \}$ are the only cases where (A1) and (A2) can be jointly fulfilled, which proves (*iii*) of the proposition.

Other equilibria with $0 < V < E$:

Note that by assuming symmetrically distributed group sizes, we can restrict our analysis to voters in i.

For such equilibria to exist, we need to show that there is some c that jointly fulfills (A1) and (A2):

$$\Phi(V, v_{j_i} = 0)\frac{1}{2} \le c \le \Phi(V, v_{j_i} = 1)\frac{1}{2}. \qquad (\mathrm{A6})$$

We give examples for $V = 1,2$ before providing a general proof that c exist that fulfill $(\mathrm{A6})$.

Example $V = 1$:

For a given $x \in [\underline{N}_i, \overline{N}_i]$, the probability that the only vote cast is in the other group, as perceived by an abstainer in i, is $\frac{E-x}{E-1} \le 1$. Then, $\Phi(1, v_{j_i} = 0) = \sum_{x=\underline{N}_i}^{\overline{N}_i} prob(x)\frac{E-x}{E-1} < \sum_{x=\underline{N}_i}^{\overline{N}_i} prob(x) = 1$. Furthermore, for the only participant, the probability that the own group has one more vote than the other is 1. Hence, $\Phi(1, v_{j_i} = 1) = 1$. Therefore we have $\Phi(1, v_{j_i} = 1)\frac{1}{2} = \frac{1}{2} > \Phi(1, v_{j_i} = 0)\frac{1}{2}$ and for any $c \in \left[\Phi(1, v_{j_i} = 0)\frac{1}{2}, \frac{1}{2}\right)$ $(\mathrm{A6})$ holds and E pure strategy Bayesian-Nash equilibria exist with exactly one voter turning out to vote.

Example $V = 2$:

From $(\mathrm{A6})$ it follows that $V = 2$ is an equilibrium outcome for any $2c \in [\Phi(2, v_{j_i} = 0), \Phi(2, v_{j_i} = 1)]$. What needs to be shown is that this set is non-empty. For a given $x \in [\underline{N}_i, \overline{N}_i]$, the probability that the two votes are divided equally across the two groups, given that j_i abstains, is $\Phi(2, v_{j_i} = 0 | x) = prob\{1 \text{ vote in} -i,\ 1 \text{ vote in } i \,|\, v_{j_i} = 0, x\}$ $= 2\frac{(E-x)(x-1)}{(E-1)(E-2)}$. Similarly, $\Phi(2, v_{j_i} = 1 | x) = \frac{E-x}{E-1}$. The set is non-empty, iff $\Delta\Phi \equiv \Phi(2, v_{j_i} = 1) - \Phi(2, v_{j_i} = 0) \ge 0$. Then, assuming for the moment $\underline{N}_i = 1$, we have $\Delta\Phi = \sum_{x=1}^{E-1} prob(x)\frac{E-x}{E-1} - 2\sum_{x=2}^{E-1} prob(x)\frac{(E-x)(x-1)}{(E-1)(E-2)}$. Since the probability distribution of x is symmetric around $x = E/2$, this gives

$$\Delta\Phi = \sum_{x=1}^{\lfloor (E-1)/2 \rfloor} prob(x)\left\{\frac{E-x}{E-1} + \frac{x}{E-1} - 2\frac{(E-x)(x-1)}{(E-1)(E-2)} - 2\frac{x(E-x-1)}{(E-1)(E-2)}\right\}$$
$$+ prob(x=1)\left\{2\frac{(E-1)(1-1)}{(E-1)(E-2)}\right\} + prob(x = E/2)\left\{\frac{E-E/2}{E-1} - 2\frac{(E-E/2)(E/2-1)}{(E-1)(E-2)}\right\},$$

or, after some rearrangements and because the last two terms of the sum are equal to zero,

$$\Delta\Phi = \sum_{x=1}^{\lfloor (E-1)/2 \rfloor} prob(x)\frac{(E-2x)^2}{(E-1)(E-2)}.$$

Next, note that the fraction is positive for $x < E/2$. Hence, $\Delta\Phi > 0$. It is easy to see that this still holds when we give up our assumption that $\underline{N}_i = 1$. $\Delta\Phi > 0$ shows that the range

$[\Phi(2, v_{j_i} = 0), \Phi(2, v_{j_i} = 1)]$ is non-empty, so all combinations of strategies yielding $V = 2$ constitute pure strategy Bayesian-Nash equilibria for $2c$ in this range.

We now turn to the general case of turnout V being an equilibrium outcome.

1) $0 < V < E$ *and* V *even*:

Similar to the argument in our examples, $(\mathrm{A6})$ is used to determine a range $[\Phi(V, v_{j_i} = 0), \Phi(V, v_{j_i} = 1)]$ in which $2c$ should lie to make V an equilibrium outcome. We then proceed to show this range is non-empty. Once again, consider group sizes x and $E - x$, $x \in \left[\max[\underline{N}_i, V/2], \min[\overline{N}_i, E - V/2]\right]$. For x outside of this range, the probability of a tie at $V/2$ is 0, and drops out of the calculation of $\Phi(\cdot)$. For given x in this range, the probability that the V votes are split equally, given that j_i abstains, is

$$\Phi(V, v_{j_i} = 0 | x) = prob\{V/2 \text{ votes in } -i, V/2 \text{ votes in } i \,|\, v_{j_i} = 0, x\}$$

$$= \binom{V}{V/2} \cdot \left\{ \frac{E-x}{E-1} \cdot \frac{E-x-1}{E-2} \cdot \ldots \cdot \frac{E-x-V/2+1}{E-V/2} \right\} \left\{ \frac{x-1}{E-V/2-1} \cdot \frac{x-2}{E-V/2-2} \cdot \ldots \cdot \frac{x-V/2}{E-V} \right\}$$

$$= \binom{V}{V/2} \cdot \frac{(E-x)\prod_{g=1}^{V/2-1}(E-x-g)(x-g)}{\prod_{h=1}^{V-1}(E-h)} \cdot \frac{x-V/2}{E-V}. \qquad (\mathrm{A7})$$

Similarly, $\Phi(V, v_{j_i} = 1 | x) = prob\{V/2 \text{ votes in } -i, V/2 \text{ votes in } i \,|\, v_{j_i} = 1, x\}$

$$= \binom{V-1}{V/2-1} \cdot \left\{ \frac{E-x}{E-1} \cdot \frac{E-x-1}{E-2} \cdot \ldots \cdot \frac{E-x-V/2+1}{E-V/2} \right\} \left\{ \frac{x-1}{E-V/2-1} \cdot \frac{x-2}{E-V/2-2} \cdot \ldots \cdot \frac{x-V/2+1}{E-V+1} \right\}$$

$$= \binom{V-1}{V/2-1} \cdot \frac{(E-x)\prod_{g=1}^{V/2-1}(E-x-g)(x-g)}{\prod_{h=1}^{V-1}(E-h)}. \qquad (\mathrm{A8})$$

Defining $\phi(V, x) \equiv \dfrac{(E-x)\prod_{g=1}^{V/2-1}(E-x-g)(x-g)}{\prod_{h=1}^{V-1}(E-h)} \geq 0$, gives

$$\Phi(V, v_{j_i} = 0 | x) = \binom{V}{V/2} \phi(V, x) \cdot \frac{x - V/2}{E - V} \quad \text{and} \quad \Phi(V, v_{j_i} = 1 | x) = \binom{V-1}{V/2-1} \phi(V, x).$$

V-equilibria exist for some c iff $\Delta\Phi \equiv \Phi(V, v_{j_i} = 1) - \Phi(V, v_{j_i} = 0) \geq 0$. Then, assuming $\underline{N}_i < V/2 + 1$ for the moment, we have

$$\Delta\Phi = \sum_{x=V/2}^{E-V/2} prob(x) \binom{V-1}{V/2-1} \phi(V,x) - \sum_{x=V/2+1}^{E-V/2} prob(x) \binom{V}{V/2} \phi(V,x) \cdot \frac{x-V/2}{E-V}$$

$$= \binom{V}{V/2} \cdot \left\{ \sum_{x=V/2}^{E-V/2} prob(x)\phi(V,x) \left[\frac{1}{2} - \frac{x-V/2}{E-V} \right] \right\}$$

$$+ prob(x=V/2) \binom{V}{V/2} \phi(V,V/2) \cdot \frac{V/2-V/2}{E-V}, \qquad (A9)$$

where we use $\binom{V-1}{V/2-1} = \frac{1}{2}\binom{V}{V/2}$. Obviously, the second term disappears since $V/2 - V/2 = 0$. This shows that assuming $\underline{N}_i < V/2+1$ so far is innocent. Now using $\max[\underline{N}_i, V/2+1]$ instead and because the distribution of x is symmetric around $x = E/2$, we have:

$$\Delta\Phi = \binom{V}{V/2} \sum_{x=\max[\underline{N}_i, V/2]}^{\lfloor (E-1)/2 \rfloor} prob(x) \left\{ \phi(V,x)\left[\frac{1}{2} - \frac{x-V/2}{E-V}\right] + \phi(V,E-x)\left[\frac{1}{2} - \frac{E-x-V/2}{E-V}\right] \right\}$$

$$+ prob(x=E/2) \binom{V}{V/2} \phi(V,E/2) \left[\frac{1}{2} - \frac{E/2-V/2}{E-V}\right]$$

$$= \binom{V}{V/2} \sum_{x=\max[\underline{N}_i, V/2]}^{\lfloor (E-1)/2 \rfloor} prob(x) \left\{ \phi(V,x) \cdot \frac{E-2x}{2(E-V)} + \phi(V,E-x) \cdot \frac{2x-E}{2(E-V)} \right\}$$

$$= \binom{V}{V/2} \sum_{x=\max[\underline{N}_i, V/2]}^{\lfloor (E-1)/2 \rfloor} prob(x) \left\{ \frac{[\phi(V,x) - \phi(V,E-x)](E-2x)}{2(E-V)} \right\} > 0, \qquad (A10)$$

because $\phi(V,x) > \phi(V,E-x)$ for $x < E/2$.

Hence, for every PU-participation game with symmetrically distributed group sizes, we can find some c such that a pure strategy Bayesian-Nash equilibrium exists in which an even number V of voters from either group participates and all others abstain, with $0 < V < E$.

2) $0 < V < E$ *and V odd*:

Similarly, we use (A6) to determine $[\Phi(V, v_{j_i} = 0), \Phi(V, v_{j_i} = 1)]$ as a range in which $2c$ should lie to make V an equilibrium outcome, and proceed to show that this range is non-empty. Consider group sizes x and $E-x$, $x \in [\max[\underline{N}_i, \lceil V/2 \rceil], \min[\overline{N}_i, E - \lfloor V/2 \rfloor]]$. Once again, if *x* is outside of the range, the probability of a tie is 0.

For given x, the probability that the V votes are split such that there is one vote less in i, given that j_i abstains, is

$$\Phi(V, v_{j_i} = 0 | x) = prob\{\lceil V/2 \rceil \text{ votes in } -i, \lfloor V/2 \rfloor \text{ votes in } i \,|\, v_{j_i} = 0, x\}$$

$$= \binom{V}{\lfloor V/2 \rfloor} \cdot \left\{ \frac{E-x}{E-1} \cdot \frac{E-x-1}{E-2} \cdot \ldots \cdot \frac{E-x-\lfloor V/2 \rfloor}{E-\lceil V/2 \rceil} \right\} \left\{ \frac{x-1}{E-\lceil V/2 \rceil - 1} \cdot \frac{x-2}{E-\lceil V/2 \rceil - 2} \cdot \ldots \cdot \frac{x-\lfloor V/2 \rfloor}{E-V} \right\}$$

$$= \binom{V}{\lfloor V/2 \rfloor} \cdot \frac{\prod_{g=0}^{\lfloor V/2 \rfloor - 1} (E-x-g)(x-g-1)}{\prod_{h=1}^{V-1} (E-h)} \cdot \frac{E-x-\lfloor V/2 \rfloor}{E-V}. \qquad \text{(A11)}$$

And, given x and that j_i participates, the probability that the V votes are split such that there is one vote more in i is

$$\Phi(V, v_{j_i} = 1 | x) = prob\{\lfloor V/2 \rfloor \text{ votes in } -i, \lceil V/2 \rceil \text{ votes in } i \,|\, v_{j_i} = 1, x\}$$

$$= \binom{V-1}{\lfloor V/2 \rfloor} \cdot \left\{ \frac{E-x}{E-1} \cdot \frac{E-x-1}{E-2} \cdot \ldots \cdot \frac{E-x-\lfloor V/2 \rfloor + 1}{E-\lceil V/2 \rceil + 1} \right\} \left\{ \frac{x-1}{E-\lceil V/2 \rceil} \cdot \frac{x-2}{E-\lceil V/2 \rceil - 1} \cdot \ldots \cdot \frac{x-\lfloor V/2 \rfloor}{E-V+1} \right\}$$

$$= \binom{V-1}{\lfloor V/2 \rfloor} \cdot \frac{\prod_{g=0}^{\lfloor V/2 \rfloor - 1} (E-x-g)(x-g-1)}{\prod_{h=1}^{V-1} (E-h)}. \qquad \text{(A12)}$$

Defining $\varphi(V,x) \equiv \dfrac{\prod_{g=0}^{\lfloor V/2 \rfloor - 1} (E-x-g)(x-g-1)}{\prod_{h=1}^{V-1} (E-h)} \geq 0$, gives

$$\Phi(V, v_{j_i} = 0 | x) = \binom{V}{\lfloor V/2 \rfloor} \varphi(V,x) \cdot \frac{E-x-\lfloor V/2 \rfloor}{E-V} \quad \text{and} \quad \Phi(V, v_{j_i} = 1 | x) = \binom{V-1}{\lfloor V/2 \rfloor} \varphi(V,x).$$

V-equilibria exist for some c iff $\Delta\Phi \equiv \Phi(V, v_{j_i} = 1) - \Phi(V, v_{j_i} = 0) \geq 0$. Then, assuming $\underline{N}_i < \lfloor V/2 \rfloor$ for the moment, we have

$$\Delta\Phi = \sum_{x=\lceil V/2 \rceil}^{E-\lfloor V/2 \rfloor} prob(x) \binom{V-1}{\lfloor V/2 \rfloor} \varphi(V,x) - \sum_{x=\lceil V/2 \rceil}^{E-\lceil V/2 \rceil} prob(x) \binom{V}{\lfloor V/2 \rfloor} \varphi(V,x) \cdot \frac{E-x-\lfloor V/2 \rfloor}{E-V}$$

$$= \binom{V}{\lfloor V/2 \rfloor} \left\{ \sum_{x=\lceil V/2 \rceil}^{E-\lceil V/2 \rceil} prob(x) \varphi(V,x) \left(\frac{1}{2} + \frac{1}{2V} - \frac{E-x-\lfloor V/2 \rfloor}{E-V} \right) \right\}$$

$$+ \, prob(x = E - \lfloor V/2 \rfloor) \binom{V-1}{\lfloor V/2 \rfloor} \varphi(V, E - \lfloor V/2 \rfloor),$$

$$=\binom{V}{\lfloor V/2 \rfloor}\left\{\sum_{x=\lceil V/2 \rceil}^{E-\lceil V/2 \rceil} prob(x)\varphi(V,x)\left(\frac{1}{2}-\frac{E-x-V/2}{E-V}+\frac{1}{2V}-\frac{1}{2(E-V)}\right)\right\}$$

$$+\ prob(x=E-\lfloor V/2 \rfloor)\binom{V-1}{\lfloor V/2 \rfloor}\varphi(V,E-\lfloor V/2 \rfloor), \qquad (\text{A13})$$

where we use $\binom{V-1}{\lfloor V/2 \rfloor}\Big/\binom{V}{\lfloor V/2 \rfloor}=\frac{1}{2}+\frac{1}{2V}$. Note that the second term of (A13) is positive and disappears if we drop the assumption $\underline{N}_i<\lfloor V/2 \rfloor$. Because the distribution of x is symmetric around $x=E/2$, we have:

$$\Delta\Phi \geq \binom{V}{\lfloor V/2 \rfloor}\sum_{x=\max[\underline{N}_i,\lceil V/2 \rceil]}^{\lfloor (E-1)/2 \rfloor} prob(x)\left\{\varphi(V,x)\left[\frac{1}{2}-\frac{E-x-V/2}{E-V}\right]+\varphi(V,E-x)\left[\frac{1}{2}-\frac{x-V/2}{E-V}\right]\right\}$$

$$+\ prob(x=E/2)\binom{V}{\lfloor V/2 \rfloor}\varphi(V,E/2)\left[\frac{1}{2}-\frac{E-E/2-V/2}{E-V}\right]$$

$$+\binom{V}{\lfloor V/2 \rfloor}\left\{\sum_{x=\lceil V/2 \rceil}^{E-\lceil V/2 \rceil} prob(x)\varphi(V,x)\left(\frac{1}{2V}-\frac{1}{2(E-V)}\right)\right\}$$

$$=\binom{V}{\lfloor V/2 \rfloor}\sum_{x=\max[\underline{N}_i,\lceil V/2 \rceil]}^{\lfloor (E-1)/2 \rfloor} prob(x)\left\{\frac{[\varphi(V,E-x)-\varphi(V,x)](E-2x)}{2(E-V)}\right\}$$

$$+\binom{V}{\lfloor V/2 \rfloor}\left\{\sum_{x=\lceil V/2 \rceil}^{E-\lceil V/2 \rceil} prob(x)\varphi(V,x)\left(\frac{1}{2V}-\frac{1}{2(E-V)}\right)\right\}. \qquad (\text{A14})$$

Note that the first term in (A14) is positive, since $\varphi(V,E-x)>\varphi(V,x)$ for $x<E/2$. But the second term is positive or zero only if $V\leq E/2$. If this is the case, then $\Delta\Phi>0$. However, if $V>E/2$, both terms in (A14) have to be evaluated, inclusive the possible second term in (A13). Whether $\Delta\Phi$ is positive or zero depends on the symmetric probability distribution at hand, which we have not specified further.

Hence, for every PU-participation game with any symmetrically distributed group sizes and $0<V\leq E/2$, there is some range for c such that a pure strategy Bayesian-Nash equilibrium exists, in which an odd number V of voters from either group participates and all others abstain. For $E>V>E/2$, these equilibria may exist, depending on the specification of the symmetric group size distribution. 1) and 2) prove (*iv*) of the proposition.

An interesting property of the V-equilibria just described is that the ranges for adjacent turnouts V are adjacent too and c is non-increasing in V. To see this, look at $c(V)_{\min}\equiv\Phi(V,v_{j_i}=0)/2$ and $c(V+1)_{\max}\equiv\Phi(V+1,v_{j_i}=1)/2$. Obviously, $c<c(V=1)_{\max}$

$=1/2$ gives the upper value. It is readily verified that $c(V|x)_{\min} = c(V+1|x)_{\max}$ holds for V even, for which we have

$$c(V|x)_{\min} = \frac{1}{2}\binom{V}{V/2}\phi(V,x)\cdot\frac{x-V/2}{E-V} = \frac{1}{2}\binom{(V+1)-1}{\lfloor (V+1)/2 \rfloor}\varphi(V+1,x) = c(V+1|x)_{\max},$$

and for V odd, for which we have (A15)

$$c(V|x)_{\min} = \frac{1}{2}\binom{V}{\lfloor V/2 \rfloor}\varphi(V,x)\cdot\frac{E-x-\lfloor V/2 \rfloor}{E-V} = \frac{1}{2}\binom{(V+1)-1}{(V+1)/2-1}\phi(V+1,x) = c(V+1|x)_{\max}.$$

Using this, it is easy to see that the difference $c(V)_{\min} - c(V+1)_{\max}$ is also equal to zero, since for V even we have

$$\sum_{x=V/2+1}^{E-V/2} prob(x)c(V|x)_{\min} - \sum_{x=\lceil (V+1)/2 \rceil}^{E-\lfloor (V+1)/2 \rfloor} prob(x)c(V+1|x)_{\max} = 0$$

and for V odd we have (A16)

$$\sum_{x=\lceil V/2 \rceil}^{E-\lceil V/2 \rceil} prob(x)c(V|x)_{\min} - \sum_{x=(V+1)/2}^{E-(V+1)/2} prob(x)c(V+1|x)_{\max} = 0.$$

Hence, we established that the ranges of possible equilibrium costs c for adjacent V are adjacent as well and that the costs are non-increasing in V.

To (*v*): Conditions (A1) to (A2) are necessary and sufficient for the existence of pure strategy Bayesian-Nash equilibria.
Q.E.D.

Proof of PROPOSITION A2 (pure strategy Bayesian-Nash equilibria in the PU-participation game with allied voters):

The proof is a straightforward extension of the proof of proposition A1. Because a floating voter knows her preference (group), she can update the probability distribution of x (group sizes), so generally: $prob(x|j_{i,a}) \neq prob(x|j_{i,f})$, except for the case E even with $prob(x=E/2|j_{i,a}) = prob(x=E/2|j_{i,f})$. Due to symmetry $prob(x|j_{i,a}) = prob(x|j_{-i,a})$ and $prob(x|j_{i,f}) = prob(x|j_{-i,f})$, hence, all allied (floating) voters have the same posterior probability distribution of group sizes. Contrary to that of floating voters, the preferences (group memberships) of allied voters are 'identifiable'. Define total aggregate participation of allied (floating) voters by $V_a \equiv V_{i,a} + V_{-i,a}$ ($V_f \equiv V_{i,f} + V_{-i,f}$), and the difference in participation between both allied groups by $\Delta V_a \equiv V_{i,a} - V_{-i,a}$.

To (i): See proof of proposition A1(i).

To (ii) and (iii): If $c < 1/2$, in order for $V = 0,1,...,E$, $V = V_a + V_f$, votes to be a pure strategy Bayesian-Nash equilibrium, no participant (non-participant) may receive a strictly higher expected payoff by deviating to abstention (participation). Then, it is necessary and sufficient for equilibria with V turnouts to exist that the following conditions hold for all allied and floating (non-)participants:

No non-participant $j_{i,a}$ and $j_{i,f}$, $i = A, B$, will change her decision if

$$\Phi(V, v_{j_{i,a}} = 0)\frac{1}{2} \le c, \quad \Phi(V, v_{j_{i,f}} = 0)\frac{1}{2} \le c, \tag{A17}$$

and no participant $j_{i,a}$ and $j_{i,f}$, $i = A, B$, will change her decision if

$$\Phi(V, v_{j_{i,a}} = 1)\frac{1}{2} \ge c, \quad \Phi(V, v_{j_{i,f}} = 1)\frac{1}{2} \ge c, \tag{A18}$$

where the probabilities $\Phi(\cdot)$ of being pivotal for allied and floating voters are similar to those in (A1) and (A2), except that we now use updated probabilities of group sizes only for floating voters (*cf.* proof of proposition A1).

Next, we establish whether and which pure strategy Bayesian-Nash equilibria exist that fulfill (A17) and (A18). We consider all possible cases $0 \le V \le E$.

Due to the binomial distribution of group sizes with $p = 0.5$ the probability of being pivotal of a floating (allied) non-participant $j_{i,f}$ ($j_{i,a}$) and an allied participant $j_{i,a}$ for $V_f - \Delta V_a$ even is given by

$$\Phi(V, v_{j_{i,f}} = 0) = \Phi(V, v_{j_{i,a}} = 0) = \Phi(V, v_{j_{i,a}} = 1)$$
$$= \begin{cases} \begin{pmatrix} V_f \\ (V_f - \Delta V_a)/2 \end{pmatrix} .5^{V_f} & if \quad V_f \ge |\Delta V_a| \\ 0 & otherwise, \end{cases} \tag{A19}$$

where ΔV_a corrects for the 'identifiable' participations of allied voters. Obviously, the strict inequalities in (A17) and (A18) cannot be fulfilled simultaneously, because floating and allied non-participants as well as allied participants have the same probability of being pivotal.

Hence, for an equilibrium to exist, it must hold that $\Phi(V, v_{j_{i,f}} = 0) = \Phi(V, v_{j_{i,a}} = 0) = \Phi(V, v_{j_{i,a}} = 1) = 2c$. We investigate whether these equalities can be fulfilled jointly with $\Phi(V, v_{j_{i,f}} = 1) \ge 2c$ for floating participants. A floating participant $j_{i,f}$'s probability of being pivotal is given by

$$\Phi(V, v_{j_{i,f}} = 1) = \begin{cases} \binom{V_f - 1}{(V_f - \Delta V_a)/2 - 1} .5^{V_f - 1} & \text{if} \quad V_f \geq |\Delta V_a| + 2 \\ 0 & \text{otherwise,} \end{cases} \tag{A20}$$

which is, given that the probability of all other voters is equal to $2c$, larger than (smaller than; equal to) $2c$ for floating participants in the group with $\Delta V_a < 0$ ($\Delta V_a > 0$; $\Delta V_a = 0$). It follows that for $V_f - \Delta V_a$ even, (A17) and (A18) being fulfilled jointly can only occur if $\Phi(V, v_{j_{i,f}} = 0) = \Phi(V, v_{j_{i,a}} = 0) = \Phi(V, v_{j_{i,a}} = 1) = 2c$ and all floating participants are members of a group with $\Delta V_a \leq 0$. For $\Delta V_a < 0$, however, since it is known that only floating voters in i participate, they are only pivotal if $V_f = \Delta V_a + 1$. But then allied participants in $-i$ would prefer to abstain. Hence, these cases cannot be equilibria. Note further that for $\Delta V_a = 0$, the trivial cases with all voters having probabilities of being pivotal equal to $2c$ occur. These equilibria are not further discussed here. No other pure strategy Bayesian-Nash equilibria exist for $V_f - \Delta V_a$ even, in which abstainers and participants coexist.

For $V_f - \Delta V_a$ odd, we can write the probability for an allied non-participant $j_{i,a}$ as

$$\Phi(V, v_{j_{i,a}} = 0) = \begin{cases} \binom{V_f}{\lfloor (V_f - \Delta V_a)/2 \rfloor} .5^{V_f} & \text{if} \quad V_f \geq |\Delta V_a| \\ 0 & \text{otherwise,} \end{cases} \tag{A21}$$

and that for an allied participant $j_{i,a}$ as

$$\Phi(V, v_{j_{i,a}} = 1) = \begin{cases} \binom{V_f}{\lceil (V_f - \Delta V_a)/2 \rceil} .5^{V_f} & \text{if} \quad V_f \geq |\Delta V_a| \\ 0 & \text{otherwise.} \end{cases} \tag{A22}$$

It is readily verified that (A21) and (A22) are equal for allied non-participants $j_{i,a}$ and allied participants $j_{-i,a}$. But the probability of allied participants $j_{i,a}$ being pivotal is larger than (smaller than; equal to) that of allied non-participants $j_{i,a}$ if $\Delta V_a > 0$ ($\Delta V_a < 0$, $\Delta V_a = 0$). It follows that allied participants and allied abstainers cannot coexist in the same group, respectively (A17) and (A18) cannot be fulfilled jointly, unless $\Delta V_a = 0$ or every allied voter in i participates and none in $-i$. Discussing $\Delta V_a = 0$ first, it is easy to see that (A21), (A22), and the probability of floating non-participants $j_{i,f}$ being pivotal are the same, or

$$\Phi(V, v_{j_{i,f}} = 0) = \Phi(V, v_{j_{i,a}} = 0) = \Phi(V, v_{j_{i,a}} = 1) = \binom{V_f}{\lfloor V_f / 2 \rfloor} .5^{V_f}, \tag{A23}$$

which is similar to $(A19)$ for $V_f - \Delta V_a$ even. As before, we investigate next whether $\Phi(V, v_{j_{i,f}} = 0) = \Phi(V, v_{j_{i,a}} = 0) = \Phi(V, v_{j_{i,a}} = 1) = 2c$ and $\Phi(V, v_{j_{i,f}} = 1) \geq 2c$ for floating participants can be fulfilled jointly for $\Delta V_a = 0$, where the probability of being pivotal of a floating participant $j_{i,f}$ is given by

$$\Phi(V, v_{j_{i,f}} = 1) = \binom{V_f - 1}{\lceil V_f / 2 \rceil - 1} .5^{V_f - 1}, \ \Delta V_a = 0 . \qquad (A24)$$

The expression in $(A24)$ is always larger than that in $(A23)$. Hence, for $V_f - \Delta V_a$ odd, pure strategy Bayesian-Nash equilibria with abstainers and participants together indeed exist for $\Phi(V, v_{j_{i,f}} = 1) > \Phi(V, v_{j_{i,f}} = 0) = \Phi(V, v_{j_{i,a}} = 0) = \Phi(V, v_{j_{i,a}} = 1) = 2c$.

With respect to the second case, it is readily verified that it cannot be an equilibrium with $\Delta V_a > 0$ and all allied voters in i participating and all allied voters in $-i$ abstaining. This is because the probability of being pivotal of required floating participants $j_{-i,f}$ is given by

$$\Phi(V, v_{j_{-i,f}} = 1) = \begin{cases} \binom{V_f - 1}{\lceil (V_f - \Delta V_a)/2 \rceil - 1} .5^{V_f - 1} & \textit{if} \quad V_f \geq |\Delta V_a| + 2 \\ 0 & \textit{otherwise}, \end{cases} \qquad (A25)$$

which is smaller than that of allied abstainers in $-i$, $\Phi(V, v_{j_{-i,a}} = 0)$. Hence, such equilibria cannot exist. No other pure strategy Bayesian-Nash equilibria exist in which abstainers and participants coexist.

This leads us to the analysis of two more possible equilibria left:

Full abstention ($V = 0$):

Full abstention cannot be an equilibrium, because any single voter can raise payoff by $1/2$ by turning out, which is larger than the participation costs ($c < 1/2$).

Full participation ($V = E$):

Equilibria with full participation exist for $V_f = F$ even {*odd*}, with $F \equiv E - 2\underline{N}_i$, as long as

$$\Phi(V = E, v_{j_{i,f}} = 1) = \binom{F-1}{F/2 - 1} .5^{F-1} = \Phi(V = E, v_{j_{i,a}} = 1) = \binom{F}{F/2} .5^{F} \geq 2c$$

$$\{ \ \Phi(V = E, v_{j_{i,f}} = 1) = \binom{F-1}{\lfloor F/2 \rfloor} .5^{F-1} > \Phi(V = E, v_{j_{i,a}} = 1) = \binom{F}{\lfloor F/2 \rfloor} .5^{F} \geq 2c \ \}. \qquad (A26)$$

To (*iii*): Conditions $(A17)$ to $(A18)$ are necessary and sufficient for the existence of pure strategy Bayesian-Nash equilibria. *Q.E.D.*