Fairly Allocating Output from a Commons

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Overview

- Key Concepts
- Exercises
- Axiomatization of Shapley Value

Key Concepts: Commons

- A <u>commons</u> is a technology used jointly by a set of agents.
  - Requires the assessment of the contributions of different agents in order to split the output.
  - Examples: musical ensemble, fishing cooperative
- Question: How can we fairly distribute the output of the commons?

Key Concepts: Shapley Value

- Shapley Value is the expected marginal cost for an agent of the commons given all possible orderings
- More formally,

$$x_i = \sum_{s=0}^{n-1} \sum_{S \in A_i(s)} \frac{s!(n-s-1)!}{n!}[C(S \cup \{ i \}) - C(S)]$$

- Number of calculations increases exponentially with number of agents
- More in a minute!

Key Concepts: Stand-alone Test

- Another answer: the stand-alone test
- First a few definitions:
- subadditive – The cost for the group of agents is less than the sum of their standalone costs.

$$C(N) \leq \sum_{i \in N} C(i)$$

- Conversely, superadditivity means that the cost for the group of agents is greater than the sum of their standalone costs

Key Concepts: Stand-alone Test

- The stand-alone test says that everyone shares the externality created by subadditivity or superadditivity
  - Formally, C subadditive $\Rightarrow x_i \le C(i)$
    C superadditive $\Rightarrow x_i \ge C(i)$
  - The Shapley value always passes the stand-alone test.
  - However, it still may be the case that a coalition of agents pay more than their stand-alone costs. This leads to…

Key Concepts: Stand-alone Core

- The stand-alone core where no coalition can pay more than its stand-alone costs
  - Formally, C subadditive $\Rightarrow \sum_{i \in S} x_i \leq C(S) \quad \forall S \subseteq N$
    C superadditive $\Rightarrow \sum_{i \in S} x_i \geq C(S) \quad \forall S \subseteq N$
  - Often known as the bargaining argument
  - Shapley value may or may not be in the core
  - Implications vary greatly with the cost function
- None of these methods for allocating the output of a commons references the willingness to pay of the agents. This idea is included in…

Key Concepts: Stand-alone surplus

- The Stand-alone surplus
  - Here surplus is denoted by the surplus function $S \rightarrow V(S)$ which assigns to each possible “virtual coalition” a value equal to their willingness to pay minus their costs.
  - You can use the other techniques (stand-alone cores, Shapley value, etc.) in allocating this surplus

Exercises

- Exercise 5.1
- Exercise 5.7
- Exercise 5.6

Axiomatization of the Shapley Value

- Equal Treatment of Equals

  If agents i and j are equal w.r.t. the cost function C, they should pay equally i.e. $\gamma_i(N, C) = \gamma_j(N, C)$.
- Irrelevance of Dummy

  An agent whose marginal cost for entering a coalition is zero for all possible coalitions should not have to pay anything. (Ex: An agent who lives at the post office.)
- Additivity

  $\gamma(N, C^1 + C^2) = \gamma(N, C^{1)\,+}\, \gamma(N, C^2)$

Axiomatization of the Shapley Value

- Marginalism

  $\partial_i C^1(S) = \partial_i C^2(S) \rightarrow \gamma_i(N, C^1) = \gamma_i(N, C^2)$

  If the marginal cost of adding agent i to coalition S with regards to cost function C^1 (say mail delivery) is equal to the marginal cost of adding i to coalition S with regards to C^2 (network cables) then i would pay the same in both cases.

Axiomatization of the Shapley Value

- Equal Impact

$$\gamma_I(N, C) - \gamma_J(N\backslash j, C^{-j}) = \gamma_J(N, C) - \gamma_i(N\backslash i, C^{-i})$$

- Potential

$$\gamma_I(N, C) = P(N, C) – P(N\backslash i, C^{-i}) \text{ for all N, I, and C}$$