How Risk Sharing May Enhance Efficiency in English Auctions[1]

Audrey Hu

Heidelberg University

Theo Offerman

University of Amsterdam/CREED

Liang Zou

University of Amsterdam/ABS

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How Risk Sharing May Enhance Efficiency in English Auctions

By Audrey Hu, Theo Offerman, and Liang Zou

We investigate the possibility of enhancing efficiency by awarding premiums to a set of highest bidders in an English auction—in a setting that extends Maskin and Riley (1984, *Econometrica* **52**: 1473-1518) in three aspects: (i) the seller can be risk averse, (ii) the bidders can have heterogeneous risk preferences, and (iii) the auction can have a binding reserve price. Our analysis reveals that the premium has an intricate joint effect on risk sharing and expected revenue, which in general benefits risk averse bidders. When the seller is more risk averse than the pivotal bidder – a condition often verifiable by deduction prior to the auction– the premium also benefits the seller and therefore leads to a Pareto improvement of the English auction. The advantage of such premium tactics is directly related to (a) the seller's degree of risk aversion, (b) the reserve price, (c) the riskiness of the object for sale, (d) the degree of heterogeneity in risk preferences among the bidders, and (e) the number of the potential bidders.

Keywords: Risk sharing, Pareto efficiency, Premium auction, English auction, Reserve price, Ensuing risk, Heterogeneous risk preferences.

JEL classification: D44

1 Introduction

Recent advances in efficient auction design brought forth a number of remarkable auction procedures that do not require seller knowledge about the bidders' private information, yet under various complex environments achieve efficient allocations of goods ex post (e.g., Dasgupta and Maskin, 2000; Jehiel and Moldovanu, 2001; Ausubel, 2004; Perry and Reny, 2002, 2005). Along the lines of the VCG mechanisms (Vickrey, 1961; Clark, 1971; Groves, 1973), all these auction procedures were judiciously designed under the assumption that the players are risk neutral or have quasilinear utility functions.

While the assumption of risk neutrality might be a good approximation of reality in some circumstances, e.g., if the bidders are firms with ample liquidity, it might be an improper fit in some other circumstances, e.g., if the bidders are consumers, business persons, or small firms with limited capital (e.g., Milgrom, 2004, p. 46). When both the seller and bidders are risk averse, an important question is whether, given an *ex post* efficient auction mechanism, the players can benefit from any risk-sharing scheme that enhances *ex ante* efficiency of the mechanism.[1,2] Put differently, can we make *all players*, i.e., the seller and all types of the prospective bidders, better off by modifying the payment rule of a mechanism while maintaining its allocation rule? As most of the auctions literature assumes risk neutrality of

[1] We use the term "ex ante" to mean the pre-auction stage when the auction rule may be subject to changes (by the seller, the auction designer, or as a result of bargaining). This may include the "interim" stage when each potential bidder has received his private information but does not know the others' information, as well as the stage when no bidder has received any private information (see, e.g., Holmstrom and Myerson, 1983; Crawford, 1985).

[2] Ex ante risk sharing should be distinguished from a separate problem of sharing *ensuing* risk between the seller and the winning bidder through joint ownership of the auctioned asset (e.g., the security design problem studied in DeMarzo, Kremer and Skrzypacz, 2005).

either the seller and/or the bidders,[3] the case for risk sharing in auctions among the risk averse seller and bidders remains unclear.[4] The main contribution of our paper is that we show how risk sharing between the seller and bidders may enhance ex ante efficiency in auctions.

We analyze the effects of risk sharing in auctions in a setting that is general in terms of the players' risk preferences. While incorporating the possibility of ensuing risk in our model, e.g., where the auctioned object is, in essence, a risky asset, we allow the players to exhibit heterogeneous risk preferences. The heterogeneity in risk preferences is a more general and realistic assumption. Apart from being commonly recognized as a stylized fact (e.g., Arrow, 1971), the assumption has been confirmed in many experimental studies following Cox, Smith and Walker (1982, 1988). For instance, Harrison, List and Towe (2007, p.437) reported that they "observe considerable individual heterogeneity in risk attitudes, such that one should not readily assume homogeneous risk preferences for the population."

For tractability, we consider a single-object auction environment and take the English auction (EA) as our benchmark auction model. The EA is perhaps the

[3]For auctions with a risk neutral seller and risk averse bidders, see, e.g., Holt (1980), Riley and Samuelson (1981), Harris and Raviv (1981), Milgrom and Weber (1982), Matthews (1983, 1987), Maskin and Riley (1984), Cox et al. (1982, 1988), Smith and Levin (1996), Eso and White (2004), and Levin and Ye (2007).

Waehrer et al. (1998) and Eso and Futo (1999) studied auctions with a risk averse seller and risk neutral bidders. Exceptions that allow both the seller and buyers to be risk averse can be found in Hu, Matthews and Zou (2010) and Hu (2011), where the focus is on the optimal reserve prices.

[4]Eso and Futo (1999) present an interesting result related to the topic of the present study. Assuming that all bidders are risk neutral and the seller is risk averse, they show that among all incentive compatible auction mechanisms there exists one that is risk-free to the seller. Consequently, this mechanism is ex ante efficient by the revenue equivalence theorem. The problem we consider is more difficult because under bidder risk aversion, the payment rule simultaneously affects the riskiness as well as the expected revenue.

most widely practiced, and the most extensively studied, auction format. Its open ascending-bid procedure ensures simplicity, transparency, optimal use of information,[5] and under various conditions leads to ex post efficient outcomes.[6] Therefore, if the EA were Pareto inefficient ex ante, it would be of practical as well as theoretical interest to find out ways to improve it. The auction format we consider is called the English premium auction (EPA), which nests the EA as a special case. The EPA proceeds just like an EA except that the highest two bidders receive a "profit share", or "premium", from the seller that is equal to a fraction α of the difference between the second and the third highest bids. Premium auctions are regularly used in Europe to sell houses, land, and machinery amongst others, and have been analyzed in several previous studies (e.g., Goeree and Offerman, 2004; Milgrom, 2004; Hu, Onderstal and Offerman, 2011; Hu, Offerman and Zou, 2011).[7]

In Hu, Offerman and Zou (2011), we studied the EPA in a symmetric interdependent values setting of Milgrom and Weber (1982) with homogeneous risk averse bidders, finding that revenue maximization is unlikely to be a good reason for the seller to offer any premiums in an EA. In the present paper, we consider instead an auction environment that extends the classical setting of Maskin and Riley (1984) in three aspects: (i) the seller can be risk averse, (ii) the bidders can exhibit heterogeneous risk preferences, and (iii) the seller can impose a binding reserve price.[8] Each

[5] See, e.g., Milgrom and Weber (1982), McAfee and McMillan (1987), McMillan (1994), Klemperer (2002), Ausubel (2004), and Perry and Reny (2005).

[6] See, e.g., Maskin (1992), Wilson (1998), Krishna (2003), Dubra Echenique and Manelli (2009), and Birulin and Izmalkov (2010).

[7] Although premium auctions have been employed under numerous different formats, they share a common characteristic that some losing bidder(s) will receive some "payback" from the seller. See, e.g., van Bochove, Boerner and Quint (2012) for a study of "Anglo-Dutch premium auctions" used in the secondary market for financial securities in the eighteenth-century.

[8] As we do not require that the seller imposes a reserve price strictly higher than his reservation

line of these extensions also generalizes our previous work (Hu, Offerman and Zou, 2011) for the special case with private values and linear premium rules. Instead of looking for optimal solutions that maximize the seller's expected revenue as done in Maskin and Riley (1984), we concentrate in this paper on the "detail free" (Wilson, 1987) EPA in order to obtain sharp and applicable results, treating the reserve price and the premium rule as exogenously given.

We obtain significant new insights into the effects of awarding premiums in auctions by these extensions: (i) the seller risk aversion is, evidently, key to understanding the risk-sharing effect of the premium tactics; (ii) for an arbitrary number (> 2) of potential bidders, the seller's expected payoff can be characterized as a functional of the utility functions of the seller and the *pivotal* bidder—the one who determines the selling price in the EA (Theorem 3); and (iii) the binding reserve price can cause the equilibrium bidding strategies in the EPA to exhibit a "jump" at the reserve price (Theorems 1-2). An important consequence of this jump bidding, a phenomenon similar to the one observed in Jehiel and Moldovanu (2000) for their second-price auction equilibrium with externalities, is that at the interim stage the same reserve price will induce the same subset of active bidders in either the EA or the EPA even when the EPA is more attractive to bidders than the EA (Theorem 5).

From the seller's viewpoint, our Theorem 3 implies that he will benefit from the premium tactics whenever he is *more risk averse than the pivotal bidder*. Although the utility function of the pivotal bidder may not be known publicly, this condition can often be assured ti hold prior to the auction by deduction. To illustrate, consider a setting in which the risk preferences of the bidders are heterogeneous and unknown to the seller. Suppose the seller himself is a businessman whose utility function

value, there is no commitment problem or loss of efficiency that may arise in the event of no sale, since the seller will then be the one who values the object the most among all players.

belongs to the same set of that of the potential bidders. Suppose further that the object for sale is a risky asset whose unknown value has a distribution common to all and that the seller chooses a reserve price insuring him against potential losses. Then, the pivotal bidder will be more risk tolerant than the seller whenever the auction concludes with a sale, given that the pivotal bidder is willing to pay more than the seller's reserve price. In such circumstances, the seller always benefits from the premium tactics—with or without knowledge about the bidders' risk preferences.

A corollary of Theorem 3 is that when bidders are risk neutral, the EPA revenue is less risky than that of the EA (Corollary 2). In Theorem 5, we show that under plausible conditions the premium tactics allow all risk averse bidders to derive higher expected utilities prior to the auction. The intuition behind this result lies in the twofold benefits that the premium offers: it reduces the average payment (Theorem 4) and it reduces the riskiness of the payment. Therefore, a general conclusion from our study is that when all players are risk averse, the EPA establishes a Pareto improvement on the EA whenever the seller is more risk averse than the *active* bidders who are willing to pay more than the reserve price (Propositions 1-2). In the cases where the seller is unable to impose a reserve price, such as in a "must sell" situation, we show that the EPA can still be attractive to the seller if he expects a sufficiently large number of heterogeneous bidders to participate in the auction (Proposition 3). In the other extreme cases where the seller knows the exact distribution of bidders' types, we show that there exists an optimal linear premium rule α^* that is strictly between 0 and 1/2 (Corollary 3).

The rest of the paper is structured as follows. In Section 2, we present the model and some preliminary results that are useful for dealing with heterogeneous risk preferences. In Section 3 we analyze the EPA equilibrium and properties of the bidding strategies. Section 4 contains the main results about the premium effects from both the seller's and bidders' perspectives. We conclude the paper in Section

5 with remarks on future research. The proofs of the lemmas and propositions are relegated to the Appendix.

2 The Model

An indivisible object is for sale via an English premium auction (EPA) to N (> 2) potential bidders. The auctioneer announces a reserve price for the object, p_0, and observes the number n of bidders who are willing to participate, i.e., to bid p_0 or above. If $n \leq 2$, then the auction will be conducted as a standard English auction (EA), where the losing bidder pays and receives nothing and the winner purchases the object for the price at which the other bidder quits. (Thus $n = 0$ results in no sale and $n = 1$ results in the only active bidder winning the object for price p_0.)

For $n > 2$, the EPA will be conducted in two stages. In the first stage, a clock price rises from p_0. At each price level, bidders decide to stay in the auction or to exit. An exit decision is irrevocable. The first stage ends when only two bidders, or *finalists*, remain active. The price level X, or *bottom price*, at which the third-to-last bidder quits will serve as a new reserve price onwards in the second stage, in which the price rises from X until one of the finalists quits. The remaining one wins the object and pays the price p at which the other finalist quits. Both finalists also receive a *premium* from the seller equal to $\alpha(p - X)$, where $\alpha \in (0, 1/2]$ is publicly known prior to the auction. Any ties are resolved randomly: in the second stage, if both finalists withdraw at the same price p, then both will receive a premium equal to $\alpha(p - X)$ and one of them will be randomly chosen to receive the object and pay the price p; in the first stage, if two or more bidders simultaneously withdraw at price X, with only one (or no) bidder left, then the auction ends like an English auction with the (randomly chosen) highest bidder paying price X for the object and no one receiving any premium.

Each potential bidder i has a private type $t_i \in [0, H] \subset \mathbb{R}$ that affects his preference for the object. Ex ante, the types t_i are independently distributed according to the same distribution function F, which has a density function $f = F'$ that is strictly positive and continuously differentiable on $(0, H]$. Given any vector of types $(t_1, ..., t_N)$, we let $t_{(1)}, t_{(2)}$ and $t_{(3)}$ denote the highest, second highest, and third highest types from among $(t_1, ..., t_N)$.

The preference of a typical bidder with type t is represented by

$$\begin{cases} w(x,t) & \text{if he wins the object and receives } x \\ u(x,t) & \text{if he loses and receives } x \end{cases} \tag{1}$$

The bidder with type t who drops out in the first stage will have utility $u(0, t)$.

We shall interpret $u(x, t)$ as type-t bidder's status-quo utility for income x. For ease of discussion, we refer to the special case where $u(x, t)$ is independent of t in (1) as the *homogeneous-utility* model (e.g., Maskin and Riley, 1984),[9] and the more general case as the *heterogeneous-utility* model in which $u(\cdot, t)$ and $u(\cdot, t')$ can be two different utility functions given any $t \neq t'$.

The functions $u(x, t)$ and $w(x, t)$ are assumed to satisfy the following conditions.[10]

A1. u and w are twice continuously differentiable.

For all $t \in [0, H]$,

A2. $w(-\infty, t) < u(0, t) < w(0, t)$.

A3. $u_1 > 0$ and $w_1 > 0$.

A4. $u(x, t) - u(z, t)$ is log-concave in $x \in (z, \infty)$, $\forall z$.

[9] The term "homogeneous utility" refers only to the fact that all *losing* bidders have the same utility for income. The *winning* bidders' utility functions can still vary with their private types.

[10] Subscripts denote the argument with respect to which a partial derivative is taken.

Condition A2 implies that all types of bidders would like to have the object if it is free, i.e., $w(0,t) > u(0,t)$; but no bidder is willing to pay too high a price for the object, i.e., $w(-\infty,t) < u(0,t)$. A3 is the usual assumption that utilities increase in income. A4 is commonly invoked to guarantee the existence of equilibria in first-price sealed-bid auctions (e.g., Athey, 2001). It holds under risk aversion but it also allows the bidders, to some extent, to be risk preferring .

The next two conditions involve the properties of the ratio[11]

$$Q(x,y,t) \equiv \frac{u(x,t) - w(x-y,t)}{u_1(x,t)} \tag{2}$$

A5. For all x, y, $Q(x,y,t)$ is decreasing in t and $-u_{11}(x,t)/u_1(x,t)$ is nonincreasing in t.

A6. For all y, t, $Q(x,y,t)$ is nonincreasing in x.

The economic interpretations of A5 and A6 will become more transparent by considering some special cases of our model. We first present a lemma that will be frequently used later on for interpretations of the main results of this paper. The lemma can be seen as a corollary of Pratt (1964, Theorem 1), which helps connect the expression in (2) to the Arrow-Pratt measure of absolute risk aversion. (See Hu, Matthews and Zou, 2012 for discussions on similar issues in an interdependent-values setting.)

Lemma 1 *Let $u, \hat{u} : \mathbb{R} \to \mathbb{R}$ be two increasing and twice continuously differentiable utility functions. Then the following conditions are equivalent, in either the strong form (indicated in brackets), or the weak form (with the bracketed material omitted):*

(i) $-u''/u' \geq -\hat{u}''/\hat{u}'$ [and $>$ for at least one x in every interval].

(ii) For all x and y such that $y \neq 0$,

$$\frac{u(x) - u(x-y)}{u'(x)} \geq [>] \frac{\hat{u}(x) - \hat{u}(x-y)}{\hat{u}'(x)}. \tag{3}$$

[11] The notational dependence of Q on the functions u and w is suppressed.

(iii) For all x and y, and for all nondegenerate random variables $\tilde{v}$ such that $E\tilde{v} < \infty$,

$$\frac{u(x) - Eu(\tilde{v} + x - y)}{u'(x)} \geq [>] \frac{\hat{u}(x) - E\hat{u}(\tilde{v} + x - y)}{\hat{u}'(x)}. \tag{4}$$

For the homogeneous-utility model, the following four special cases have been considered in Maskin and Riley (1984) where U is an increasing von Neumann-Morgenstern utility function.

Case 1 $w(x,t) = U(t + x)$ and $u(x,t) \equiv U(x)$.

Case 2 $w(x,t) = U(t + \psi(x))$ and $u(x,t) \equiv U(\psi(x))$, where $\psi' > 0$, $\psi'' \leq 0$, and $\psi(0) = 0$.

Case 3 $w(x,t) = \int U(v + x) dK(v|t)$ and $u(x,t) \equiv U(x)$, where $K(v|t) > K(v|\hat{t})$ for all $t < \hat{t}$.

Case 4 $w(x,t) = (1 + t)U(t + x)$ and $u(x,t) \equiv U(x)$, where $U \geq 0$.

Case 1 is the standard private-values model. Case 2 allows a bidder to assign certain quality to the auctioned object, which may not have an equivalent monetary value. Case 3 allows the object to entail ensuing risks, where the distribution of the uncertain monetary value v of the object depends on a bidder's type. The conditional distribution of v for a higher type exhibits first-order stochastic dominance over that for a lower type. Case 4 provides an example in which winning the object gives one a greater ability to derive pleasure, crudely translated into a higher marginal utility as well as utility for income.[12]

Obviously, the conditions A1-A4 are easily satisfied with proper assumptions on U and K for all the four cases.[13] The following lemma relates U to A5 and A6.

[12] See Maskin and Riley (1984) for more detailed discussions of these cases.

[13] For instance, for Case 2 u is log-concave as long as U is log-concave in the sense of A4, since ψ is a (weakly) concave function.

Lemma 2 *For Cases 1-4, $U' > 0$ implies A5. If U exhibits nonincreasing absolute risk aversion, then A6 holds for Cases 1-3. If in addition U is nonnegative and is log-concave, then A6 holds for Case 4.*

For the heterogeneous-utility model, it is clear that each of the Cases 1-4 can be generalized straightforwardly by replacing $U(x)$ with $U(x,t)$, so that $u(x,t) = U(x,t)$ (or $U(\psi(x),t)$ for Case 2').

Case 1' $w(x,t) = U(v(t) + x, t)$ where v is twice continuously differentiable with $v, v' > 0$.

Case 2' $w(x,t) = U(v(t) + \psi(x), t)$ where $\psi' > 0$, $\psi'' \leq 0$, and $\psi(0) = 0$.

Case 3' $w(x,t) = \int U(v + x, t) dK(v|t)$ with $K(v|t) \geq K(v|\hat{t})$ for all $t < \hat{t}$.

Case 4' $w(x,t) = (1+t)U(v(t) + x, t)$ where $U \geq 0$.

Note that these cases also generalize Cases 1-4 with respect to some other details. For instance, Case 3' allows the distribution K to be independent of t so that all bidders have the same probability distribution over v. This can be a situation in which all available information has been "priced" into the object for sale but because the bidders have different risk attitudes they may still have different expected (utility) payoffs upon winning. More generally, Case 3' allows the bidders' types to affect their risk preferences as well as their expectations about the object's uncertain value. For instance, t may be correlated to a bidder's wealth, a higher wealth level giving the bidder more favorable conditions for using or deriving values from the object.

The next lemma shows the more general conditions on $U(x,t)$ that are sufficient for A1-A6 to hold.

Lemma 3 *Assume that $U(x,t)$ is twice continuously differentiable with $U_1 > 0$, that $U(0,t) < \int U(v,t) dK(v|t)$ for all t, and that $U(x,t) - U(z,t)$ is log-concave in*

$x \in (z, \infty)$ for all z. Further assume that $U(\cdot, t)$ exhibits nonincreasing absolute risk aversion and $U(\cdot, t)$ is more risk averse than $U(\cdot, \hat{t})$ whenever $t < \hat{t}$. Then A1-A6 hold for Cases 1'-4'.

Indeed, in all these cases the conditions A5 and A6 can be replaced by a joint condition that $-u_{11}(x,t)/u_1(x,t)$ is decreasing in t and nonincreasing in x. An important special case is where $u(x,t)$ exhibits CARA in x for all t, or that A6 holds with $Q_1 = 0$. If A6 holds with $Q_1 < 0$, then by Lemma 1 it corresponds to the cases in which $u(x,t)$ exhibits DARA in x for all t. However, since these cases are just special examples of our model and the function $w(x,t)$ can be given other forms or interpretations (e.g., non-expected utility preferences), we maintain A5 and A6 in this paper for generality.

3 Equilibrium

We begin with the EA equilibrium, and then proceed with a detailed analysis of the EPA equilibrium in this section. The EA equilibrium is useful as a benchmark for analyzing the premium effects in the next section.

3.1 English auction

In the EA, it is routine to check that there exists a unique symmetric equilibrium in our setting. In this equilibrium, it is a (weakly) dominant strategy for a type-t bidder to stay in the auction until the price reaches $\eta(t)$ such that

$$w(-\eta(t), t) = u(0, t) \tag{5}$$

By assumptions A1-A3 the bid function η is well defined on $[0, H]$, and it is increasing by A5.

If the reserve price $p_0 < \eta(0)$, it has no effect and all bidders will participate in the EA. If $p_0 > \eta(H)$ then no bidder will be interested in bidding. From now on we assume that $p_0 \in [\eta(0), \eta(H)]$. Then, there exists a *screening level* $t_0 \in [0, H]$ defined by $\eta(t_0) = p_0$. A bidder will abstain from bidding in the EA if and only if his type is lower than t_0.

An important property of the EA equilibrium is that the object for sale will be allocated to the one with the highest type. This is the one who has the highest willingness to pay for the object and therefore the EA is *ex post efficient*. It is important to note that conditions A5-A6 imply that the winning bidder in the EA should be (weakly) more risk tolerant than all other bidders. This is to be expected in our model even where all bidders have the same status-quo utility function, because by the wealth effect, the property of DARA (CARA) automatically implies that the one who values the object most is (weakly) more risk tolerant than all other bidders. This result is further strengthened under A5 for the more general environments in which the object for sale entails ensuing risk.

3.2 English premium auction

If the seller chooses the same reserve price p_0 for the EPA, we show in Theorem 2 that the reserve price will induce the same screening level t_0 so that the subsets of active types who are willing to participate in the EA and the EPA are the same. According to the EPA rule, if there are two or fewer bidders staying in the auction at price p_0, then the auction reduces to the EA and the preceding analysis of equilibrium strategy $\eta(t)$ holds for this special case.

Now suppose that there are $n > 2$ bidders willing to participate in the EPA under the reserve price p_0. By backward induction, suppose that the first stage has ended with bottom price $X \geq p_0$. As in the EA, we focus on symmetric equilibria in which both finalists adopt the same bid function $b(\cdot, X) : [0, H] \to [X, \infty)$. For a finalist with type t, $b(t, X)$ specifies the price at which he will quit. We say that b is a *second-stage EPA symmetric equilibrium* (or second-stage equilibrium for short) if conditional on X, adopting $b(\cdot, X)$ maximizes each finalist's expected utility given that the other finalist adopts the same strategy b.

Because the active bidders in the first stage compete to enter the second stage, the anticipated second-stage equilibrium will affect their first-stage strategies. As will be verified, the equilibrium bid function $b(t, X)$ has the property of being continuously differentiable, increasing in t and nondecreasing in X. These properties allow us to define (implicitly) the first-stage strategy β induced from b by

$$\beta(t) = b(t, \beta(t)) \tag{6}$$

We say that $b^* = (\eta, \beta, b)$ is an *EPA symmetric equilibrium* (or EPA equilibrium for short) if (i) $b(\cdot, X)$ is a second-stage equilibrium conditional on any bottom price X; (ii) in the first stage with $n > 2$ active bidders, conditional on any updated information it is optimal for each bidder to adopt strategy β providing the other bidders adopt β; (iii) in the first stage with $n = 2$ active bidders, it is a (weakly) dominant strategy for each bidder to adopt strategy η; and (iv) prior to the auction a type-t bidder chooses to stay at price p_0 if and only if his expected payoff from the subsequent auction game is no less than $u(0, t)$.

Now, by backward induction, suppose that the bottom price X is known from the first stage and that the two finalists both have types in $[r, H]$ for some $r \geq 0$. Let $G(t) = (F(t) - F(r)) / (1 - F(r))$ denote the conditional probability that the opponent of a finalist has type $y \in [r, t]$. Suppose the opponent of this finalist chooses strategy $b(\cdot, X)$. The auction game requires that $b \geq X$ and we assume that b is uniformly bounded from above by some $M < \infty$ (e.g., no bidder has unlimited wealth).[14]

[14]This assumption is in fact inessential; it sufficies to directly assume that the bidders' objective

Given his own type t and the opponent's strategy $b(\cdot, X)$, the problem of the finalist is to choose a bid $b \in [X, M]$ that maximizes his expected utility[15]

$$U(b, t|X) = \int_{b(y,X)<b} w\left(\alpha(b(y, X) - X) - b(y, X), t\right) dG(y)$$
$$+u\left(\alpha(b - X), t\right) \int_{b(y,X)>b} dG(y)$$
$$+\frac{1}{2}\left[w\left(\alpha(b - X) - b, t\right) + u\left(\alpha(b - X), t\right)\right] \int_{b(y,X)=b} dG(y)$$

where the first term on the right-hand side of the equation results from the event of winning, the second term from the event of losing, and the last term from the event where there is a tie at b. The strategy $b(\cdot, X)$ is a second-stage symmetric equilibrium if and only if $b(t, X)$ is the best response strategy for the bidder with type t, i.e.,

$$U(b(t, X), t|X) \geq U(b, t|X) \quad \forall t \in [r, H],\ \forall b \in [X, M].$$

Our first theorem provides the characterization of a second-stage EPA symmetric equilibrium, and demonstrates the existence and uniqueness of such an equilibrium.

As a preparation, define $B(H, X)$ to be the bid that makes type-H bidder indifferent between winning and losing, given X. It is the solution B that solves

$$u(\alpha(B - X), H) = w(\alpha(B - X) - B, H). \tag{7}$$

Since the left side in (7) increases in B and the right side decreases in B, by A1-A3 the solution $B = B(H, X)$ is uniquely defined and differentiable in X.

Theorem 1 *Suppose A1-A5 hold. Suppose the first stage EPA attracted $n > 2$ active bidders and ended with a bottom price $X \in [p_0, B(H, H))$. Then there exists*

functions exist given the opponents' strategies (see footnote 15).

[15] The Lebesgue integrals exist because $b(y, X)$ is bounded and w and u are continuous.

a unique second-stage equilibrium

$$b(\cdot, X) : [r, H] \to [X, B(H, X)]$$

for some $r \in [0, H)$*. The function* $b(\cdot, X)$ *is the solution of the differential equation in (8) under the boundary condition in (9):*

$$b_1(t, X) = \frac{1}{\alpha} \frac{u\left(\alpha(b - X), t\right) - w\left(\alpha(b - X) - b, t\right)}{u_1\left(\alpha(b - X), t\right)} \frac{f(t)}{1 - F(t)}, \quad t \in [r, H) \tag{8}$$

$$b(H, X) = B(H, X) \tag{9}$$

such that $b_1(\cdot, X) > 0$ *on* $[r, H)$*.*

The proof of this theorem follows essentially the same steps as in Hu, Offerman and Zou (2011, Theorem 1), hence it is omitted. Note that in the statement of this theorem, r is unspecified. This will be endogenously derived from the first-stage bidding equilibrium, which implies $b(r, X) = X$. Therefore the uniqueness of $b(\cdot, X)$ is meant to hold in the sense that there exists an r such that b is uniquely defined on $[r, H]$.

The following lemma shows that A6 implies that the second-stage equilibrium $b(t, X)$ is nonincreasing in the bottom price on its *effective domain:*

$$\Omega = \{(t, X) \in [0, H] \times [p_0, B(H, H)] : X \leq b(t, X) \leq B(H, X)\}.$$

Lemma 4 *Assume A1-A6. Then the second-stage equilibrium strategy* $b(t, X)$ *satisfies on its effective domain (i)* $b_2 = 0$ *if* $Q_1 = 0$*, and (ii)* $b_2 < 0$ *if* $Q_1 < 0$*.*

To complete the equilibrium analysis, we now turn to the bidders' first-stage strategy β defined in (6). By Lemma 4 and the implicit function theorem, $\beta(t)$ is well defined and is continuously differentiable, satisfying

$$\beta'(t) = \frac{b_1(t, \beta(t))}{1 - b_2(t, \beta(t))} \tag{10}$$

The next lemma establishes the intuitive property that the bids are in general higher in the EPA than in the EA.

Lemma 5 *A1-A6 imply that* $\beta(t) > \eta(t)$ *for all* $t \in [t_0, H)$ *and* $t_0 \in [r, H)$.

Because b is increasing, ties can be ignored as they occur only with a zero probability. In equilibrium, a finalist with type $t \geq r$ will have a conditional expected utility equal to

$$
\begin{aligned}
U^*(t|r, X) &= \frac{1}{1-F(r)} \int_r^t w\left(\alpha(b(y, X) - X) - b(y, X), t\right) dF(y) \\
&\quad + \frac{1-F(t)}{1-F(r)} u\left(\alpha(b(t, X) - X), t\right) \qquad (11)
\end{aligned}
$$

Given reserve price p_0, the EPA strategy b^* can now be described as follows. For all N potential bidders,

(1) pre-auction strategy: $\left\langle \begin{array}{lll} b^*(t) \geq p_0 & \text{for} & t \geq t_0 \\ b^*(t) < p_0 & \text{for} & t < t_0 \end{array} \right.$

This means that bidders with types lower than t_0 choose to abstain from bidding and the rest choose to participate. Once the auction begins, it becomes common knowledge how many bidders have types $t \geq t_0$. Thus for these active bidders,

(2) first-stage strategy: $\left\langle \begin{array}{lll} b^*(t) = \beta(t) & \text{if} & n > 2 \\ b^*(t) = \eta(t) & \text{if} & n \leq 2 \end{array} \right.$

The case with $n \leq 2$ is straightforward, as the auction becomes an EA and will end as soon as only one bidder remains. For $n > 2$, the first stage will end with a bottom price $\beta(t_{(3)})$ and by $\beta' > 0$ it becomes common knowledge that the two finalists both have types in $[t_{(3)}, H]$. Hence for the two finalists,

(3) second-stage strategy: $b^*(t) = b(t, \beta(t_{(3)}))$.

By strategy b^*, the same reserve price p_0 in the EA and the EPA induce the same screening level t_0. But unlike the EA equilibrium in which $\eta(t_0) = p_0$, for $n > 2$ the bid function in the EPA has a "jump" at t_0 because $\beta(t_0) > \eta(t_0) = p_0$ (by Lemma 5). In other words, there will be no bid in the price interval $(p_0, \beta(t_0)]$. This jump bidding is caused by the uncertainty at the pre-auction stage about the number n of active participants who will stay at the reserve price p_0. In principle, the premium would induce all types of bidders to bid higher in the EPA than in the EA. But the premium is effective if and only if $n > 2$. As shown in the proof of Theorem 2, given the equilibrium strategies of other bidders, a bidder with type lower than the screening level t_0 will not gain by being an active participant in either the scenario $n > 2$ or the scenario $n \leq 2$. Hence it is optimal for him to abstain from bidding. On the other hand, all bidders with types higher than t_0 will find it optimal to bid as though the premium is effective. Hence the jump of b^* at t_0.[16]

Our next theorem establishes that b^* is indeed an equilibrium.

Theorem 2 *Suppose A1-A6 hold.*[17] *Suppose the reserve price is p_0. Then the strategy b^* constitutes an EPA equilibrium.*

Proof. If only two or fewer bidders are interested in purchasing the object for p_0, the situation reduces to the EA and η is a dominant strategy by standard arguments.

Now suppose at the start of the first stage there are $n \geq 3$ active bidders. We analyze b^* by backward induction. Theorem 1 has established $b(t, X)$ to be a

[16] Jehiel and Moldovanu (2000) derive a similar jump-bidding property in their second-price auction equilibrium with negative externalities. In their model with two bidders and a binding reserve price, a subset of types with private values lower than the reserve price face the uncertainly whether the opponent will bid higher or lower than the reserve price (similar to our $n > 2$ or $n \leq 2$ scenarios). By deduction, under both scenarios these low-value types will not stand to gain and therefore will bid zero. As the reserve price does not affect the equilibrium bids by other types, which are higher than their true values due to the externality, the jump bidding occurs at the level of the reserve price.

Milgrom and Weber (1982) also observe a similar "jump" property in their analysis of second-price equilibrium with interdependent values and a binding reserve price. For related analyses see also, e.g., Jehiel and Moldovanu (1996) and Caillaud and Jehiel (1998).

[17] The assumption A6 simplifies the proof of this theorem but it is unnecessary for the result (cf. Hu, Offerman and Zou, 2011, Theorem 2). It will be mainly needed to establish Theorem 3 and the subsequent results in the next section.

unique second-stage equilibrium given any bottom price X, which induces a unique first-stage strategy $\beta(t) = b(t, \beta(t))$.

First stage: [18]

Let p denote the ongoing price that rises from p_0 and define $r(p) = \beta^{-1}(p)$. Consider one of the active bidders with type t and suppose that the other bidders follow the strategy β.

Clearly, as long as $\beta(t) > p$ so that $t > r(p)$, it is optimal for the bidder to stay because $U^*(t|r(p), p) > U^*(r(p)|r(p), p) = u(0, t)$, as can be seen by substituting $r(p)$ for r and p for X in (11).

If the bidder quits at a price $p > \beta(t)$ and does not enter the second stage, his utility is $u(0, t)$. However, if the bidder stays after the price exceeds $\beta(t)$ and becomes a finalist at some bottom price $X > \beta(t)$, he must then bid in the second stage as if his type was higher than t. Given that X is determined by a bidder who has a type higher than t by following strategy β, the type-t finalist will have to quit at X in the second stage. This is because his marginal cost of bidding higher than X in the second stage necessarily exceeds the marginal benefit. But since bidding X yields $u(0, t)$, it is a (weakly) dominant strategy to quit earlier in the first stage. The reason is that in case the other finalist also bids the bottom price, the random allocation of the object to the bidder will incur a loss, without any compensating premium.

Pre-auction stage:

Consider first a bidder with type $t < t_0$. Prior to the auction, he is unsure about the number n of bidders who will bid at p_0. We show that it is optimal for the bidder to abstain from bidding. If he stays at p_0, he faces three possible scenarios.

[18] The proof of the first-stage equilibrium is analogous to Hu et al. (2011, Theorem 2) for the homogeneous-utility model with a more general information structure. We sketch a proof here for completeness. The proof of the pre-auction stage equilibrium is new.

(i) No other bidder stays at p_0. In this case $n = 1$ and by (5) the bidder purchases the object at a loss. (ii) Only one other bidder stays at p_0. In this case $n = 2$ and the English auction policy implies that the bidder will have no expected profit to be made. He has to quit immediately or else face a potential loss should he become the winner. (iii) There are $n \geq 3$ bidders staying at p_0 (including this one with type $t < t_0$). Then, given that the other bidders will adopt strategy β in the first stage, and given that these bidders have followed the pre-auction strategy so that their types are no less than t_0, the bidder with $t < t_0$ will have no chance to become a finalist unless he deviates from strategy β. This is suboptimal, however, as it is optimal to follow strategy β in the first-stage.

Consider next a bidder with type $t \geq t_0$. It is clear that in all the above possible scenarios (i)-(iii), he will have an expected payoff higher than (if $t > t_0$) or equal to (if $t = t_0$) his status-quo utility $u(0, t)$. Therefore, it is optimal for the bidder to participate in the auction.

We conclude that b^* is an EPA equilibrium. ■

4 The Premium Effects

We now turn to investigating the welfare implications of rewarding premiums, from the seller's perspective first, and then from that of the bidders.

4.1 Seller's perspective

Suppose the seller's utility function, V, is twice differentiable and that the seller has a certainty equivalent value for the object equal to $v_0 \leq p_0$.[19]

Let $f_{(2)}^N$ denote the density function of the second-highest type $t_{(2)}$, with the

[19] In our setting, if the seller were able to choose the reserve price optimally for the EA, then $p_0 > v_0$ (e.g., Hu et al., 2010).

associated cumulative distribution $F^N_{(2)}$. The seller's expected utility in the EA can then be written as

$$
\begin{aligned}
& V_N(p_0|\text{EA}) \\
= \; & V(v_0)F(t_0)^N + V(p_0)NF(t_0)^{N-1}(1-F(t_0)) + \int_{t_0}^{H} V\left(\eta(y)\right) dF^N_{(2)}(y)
\end{aligned}
$$

where the first term on the right-hand side comes from the event of $t_{(1)} \leq t_0$, the second term from event $t_{(2)} \leq t_0 < t_{(1)}$, and the last term from the event $t_0 < t_{(2)}$. We call the bidder of type $t_{(2)}$ the *pivotal* bidder, who determines the selling price in the EA.

In the EPA, suppose the seller chooses the same reserve price p_0. Let $f^N_{(2)(3)}$ denote the joint density of the second- and the third-highest types. The seller's expected utility in the EPA is then given by

$$
\begin{aligned}
V_N(\alpha, p_0|\text{EPA}) \;=\; & V(v_0)F(t_0)^N + V(p_0)NF(t_0)^{N-1}(1-F(t_0)) \\
& + \int_0^{t_0}\int_{t_0}^{H} V\left(\eta(y)\right) f^N_{(2)(3)}(y,z)dydz && (12) \\
& + \int_{t_0}^{H}\int_{z}^{H} V\left(R(y,\beta(z))\right) f^N_{(2)(3)}(y,z)dydz && (13)
\end{aligned}
$$

where the term in (12) comes from the event $t_{(3)} < t_0 < t_{(2)}$, the term in (13) from the event $t_0 \leq t_{(3)}$, and

$$
R(y,\beta(z)) \equiv b(y,\beta(z)) - 2\alpha\left(b(y,\beta(z)) - \beta(z)\right) \tag{14}
$$

is the seller's revenue conditional on $t_{(2)} = y$ and $t_{(3)} = z \geq t_0$.

The next theorem provides a key result concerning the lower bound for the difference between the seller's expected payoffs in the EPA and the EA. This bound is "tight" in that it is reached under condition (i) in the theorem.

Theorem 3 *Assume A1-A6 and that $V'' \leq 0$. Then*

$$\begin{aligned} & V_N(\alpha, p_0|EPA) - V_N(p_0|EA) \\ \geq \; & \int_{t_0}^{H} \Phi(t) \left(1 - \left(\frac{F(t_0)}{F(t)}\right)^{N-2}\right) V'\left(\beta(t)\right) dF_{(2)}^{N}(t) \qquad (15) \\ \text{where } \Phi(t) \; = \; & \frac{V\left(\beta(t)\right) - V(\eta(t))}{V'\left(\beta(t)\right)} - \frac{u\left(0,t\right) - w\left(-\beta(t),t\right)}{u_1\left(0,t\right)} \qquad (16) \end{aligned}$$

The inequality in (15) is (i) an equality if $Q_1 = 0$ and either $V'' = 0$ or $\alpha = 0.5$, and (ii) is a strict inequality if $Q_1 < 0$, or if $V'' < 0$ and $\alpha < 0.5$.

Proof. The difference between the seller's expected payoffs in the EPA and the EA is uniquely determined by their difference in the event $t_0 \leq t_{(3)}$. Therefore

$$\begin{aligned} & V_N(\alpha, p_0|\text{EPA}) - V_N(p_0|\text{EA}) \\ = \; & \int_{t_0}^{H} \int_{z}^{H} \left[V\left(R(y, \beta(z))\right) - V\left(\eta(y)\right)\right] f_{(2)(3)}^{N}(y,z) dy dz \end{aligned}$$

Substituting $f_{(2)(3)}^{N}(y,z) = N(N-1)(N-2)F(z)^{N-3}(1-F(y))f(z)f(y)$ gives

$$\begin{aligned} & V_N(\alpha, p_0|\text{EPA}) - V_N(p_0|\text{EA}) \\ = \; & N(N-1) \int_{t_0}^{H} \left(\int_{z}^{H} \left[V\left(R(y, \beta(z))\right) - V\left(\eta(y)\right)\right] (1-F(y)) dF(y)\right) dF(z)^{N-2} \end{aligned}$$

Integrating by parts, and noting that $R(z, \beta(z)) = \beta(z)$, we obtain

$$\begin{aligned} & V_N(\alpha, p_0|\text{EPA}) - V_N(p_0|\text{EA}) \\ = \; & -N(N-1) \\ & \times \int_{t_0}^{H} \left[F(z)^{N-2} - F(t_0)^{N-2}\right] \frac{\partial}{\partial z} \int_{z}^{H} \left[V\left(R(y, \beta(z))\right) - V\left(\eta(y)\right)\right] (1-F(y)) dF(y) dz \\ = \; & N(N-1) \int_{t_0}^{H} \left[F(z)^{N-2} - F(t_0)^{N-2}\right] \left[V\left(\beta(z)\right) - V\left(\eta(z)\right)\right] (1-F(z)) dF(z) \\ & -N(N-1) \int_{t_0}^{H} \left[F(z)^{N-2} - F(t_0)^{N-2}\right] \int_{z}^{H} \frac{\partial}{\partial z} V\left(R(y, \beta(z))\right) (1-F(y)) dF(y) dz \qquad (17) \end{aligned}$$

The partial derivative

$$\frac{\partial}{\partial z} V\left(R(y, \beta(z))\right) = V'\left(R(y, \beta(z))\right) R_2(y, \beta(z)) \beta'(z)$$

where $\beta' = b_1/(1-b_2)$ from (10) and $R_2 = 2\alpha + (1-2\alpha)b_2$ by (14). Since $b_2 \leq 0$ by Lemma 4, we have $R_2 \leq 2\alpha$ and $\beta'(z) \leq b_1(z, \beta(z))$. Since $R_1 = (1-2\alpha)b_1 \geq 0$, $V'' \leq 0$ implies $V'(R(y, \beta(z))) \leq V'(\beta(z))$. Consequently,

$$\frac{\partial}{\partial z} V(R(y, \beta(z))) \leq V'(\beta(z))\, 2\alpha b_1(z, \beta(z)) \tag{18}$$

It follows that

$$\begin{aligned}
& \int_z^H \frac{\partial}{\partial z} V(R(y, \beta(z)))\,(1-F(y))dF(y) \\
\leq \;& V'(\beta(z))\, 2\alpha b_1(z, \beta(z)) \left(\int_z^H (1-F(y))dF(y) \right) \\
= \;& V'(\beta(z))\, \alpha b_1(z, \beta(z))(1-F(z))^2 \\
= \;& V'(\beta(z))\, \frac{u(0,z) - w(-\beta(z), z)}{u_1(0,z)} (1-F(z)) f(z)
\end{aligned}$$

where we used (8) to obtain the last equation. Substituting this inequality into (17), rearranging terms, and changing the notation of variable z to t, we obtain (15)-(16).

For the special case where $Q_1 = 0$, by Lemma 4(i) we have $b_2 = 0$ and therefore $R_2 = 2\alpha$ and $\beta'(z) = b_1(z, \beta(z))$. If, in addition, either $V'' = 0$ or $\alpha = 0.5$, then the inequality in (18) holds as an equality by the fact that $R(y, X) \equiv X$ for $\alpha = 0.5$. The same deduction will then yield (15) as an equation.

If $V'' < 0$ and $\alpha < 0.5$, then by the fact that $R(y, \beta(z))$ is an increasing function of y the inequality in (18) holds strictly. This is also true with $Q_1 < 0$, since by Lemma 4(ii), $b_2 < 0$ implies $R_2 < 2\alpha$ and $\beta'(z) < b_1(z, \beta(z))$. The subsequent deduction will then leads to a strict inequality in (15). ■

By inspecting (15), we find that the relative performance of the EPA from the seller's perspective depends only on the distribution of the second-highest type $F_{(2)}^N$, where $(u(\cdot, t), w(\cdot, t))$ in (16) stands for the preference functions of the pivotal bidder in the EA (i.e., $t = t_{(2)}$). Therefore, a sufficient condition for the EPA to outperform the EA is that the function $\Phi(t)$ in (16) is positive for all $t \in (t_0, H]$. In light of a result in Hu, Offerman and Zou (2011) that the premium lowers expected revenue when bidders are risk averse (see also Theorem 4 in the next subsection), Theorem 3 suggests a strong risk sharing effect of the premium: even though the expected revenue is lower, the seller may strictly prefer the EPA for the reduction of revenue risk.

It is instructive to use Case 1' as an example and see how the sign of $\Phi(t)$ can be determined. For Case 1', $w(x,t) = u(v(t)+x,t)$. So by (5), $w(-\eta(t),t) = u(0,t)$ implies $\eta(t) = v(t)$. Now assume that

$$-\frac{V''(x)}{V'(x)} \geq -\frac{u_{11}(y,t_0)}{u_1(y,t_0)}, \quad \forall x,y \in \mathbb{R} \tag{19}$$

Then, by Lemma 1

$$\frac{V\left(\beta(t)\right) - V(\eta(t))}{V'\left(\beta(t)\right)} \geq \frac{u\left(x+\beta(t),t_0\right) - u(x+\eta(t),t_0)}{u_1\left(x+\beta(t),t_0\right)} \qquad \forall x$$

In particular, for $x = -\beta(t)$ we have

$$\begin{aligned} & \frac{V\left(\beta(t)\right) - V(\eta(t))}{V'\left(\beta(t)\right)} \geq \frac{u\left(0,t_0\right) - u(\eta(t)-\beta(t),t_0)}{u_1\left(0,t_0\right)} \\ > \; & \frac{u\left(0,t\right) - u(\eta(t)-\beta(t),t)}{u_1\left(0,t\right)} \; \forall t > t_0, \text{ by A5 and Lemma 1} \\ = \; & \frac{u\left(0,t\right) - w(-\beta(t),t)}{u_1\left(0,t\right)} \text{ by } \eta(t) = v(t) \end{aligned} \tag{20}$$

This shows $\Phi(t) > 0$ and therefore $V_N(\alpha, p_0|\text{EPA}) > V_N(p_0|\text{EA})$.

The condition (19) means that regardless of the respective income levels, the seller is more risk averse than the type-t_0 bidder. This condition removes the "wealth effect" that may cause ambiguity in comparing relative risk aversion between individuals at different wealth levels. Indeed, in general, the bidders' degrees of risk aversion as being modelled depend on how we "normalize" their status quo wealth. This has been assumed to be zero in our model by convention. Such a normalization is innocuous if the bidders exhibit CARA, but in the case of DARA it makes the bidders "appear" to be more risk averse than they actually are – given the supposition that each bidder has sufficient funds to purchase the object for sale. To avoid such ambiguities, we therefore invoke the assumption $Q_1 = 0$ for the following propositions. This assumption is akin to the CARA assumption used elsewhere in auction theory (e.g., Milgrom and Weber, 1982; Matthews, 1983) as well as other fields of studies.[20]

Proposition 1 *For Cases 1'- 4', suppose A1-A6 hold. Suppose* $Q_1 = 0$ *for Cases 2'-4'. Then (19) implies that a risk averse seller with reserve price* p_0 *has a higher expected utility in the EPA, given any premium rule* $\alpha \in (0, 1/2)$*, than in the EA.*

The next proposition concerns the situation in which the seller's preference belongs to the same population of the bidders.

Proposition 2 *Suppose A1-A5 hold and* $Q_1 = 0$*. Suppose that the seller's preference is the same as a bidder with type* $t_0 \in [0, H)$*, and that* $u(p_0, t_0) = w(0, t_0)$ *(the seller's status-quo utility if there is no sale). Assume that* $u_{11}(\cdot, t_0) \leq 0$ *and the seller chooses reserve price* p_0*. Then* $V_N(\alpha, p_0|EPA) > V_N(p_0|EA)$ *for all* $\alpha \in (0, 1/2)$.

This proposition shows how the reserve price can be used to deduce that the seller is always better off by employing the EPA rather than the EA.

Our next proposition concerns the effect of bidder number N prior to the auction.

Proposition 3 *Suppose that the assumptions of either Proposition 1 or Proposition 2 hold, except that the seller does not impose a reserve price (i.e., reserve price*

[20] We agree with Milgrom (2004, p. 93-94) that using CARA is an analytical technique, and it by no means prejudges the importance of wealth effects. Of course, alternatively, the conclusions of these propositions can be arrived at by simply assuming that the bidders exhibit DARA and are sufficiently more wealthy than the seller. But such "manipulations" will not add any new insight.

equal to zero). Then, for all $\alpha \in (0, 1/2)$, there exists a number $N_\alpha > 2$ such that $V_N(\alpha, 0|EPA) > V_N(0|EA)$ for all $N > N_\alpha$.

It can be easily seen from the proof of this proposition that the result holds for any arbitrary reserve price. A higher reserve price tends to lower the threshold number N_α of the bidders.

An immediate corollary concerning the expected revenue of Propositions 1-3 is as follows.

Corollary 1 *Suppose the bidder population includes a risk neutral type, say, $t_0 \in [0, H)$. Then, under A1-A6, for arbitrary reserve price p, (i) $p \geq p_0$ implies that the expected revenue in the EPA is greater than that in the EA; and (ii) $p < p_0$ implies that for all $\alpha \in (0, 1/2)$, there exists an $N_\alpha > 2$ such that the expected revenue in the EPA is greater than that in the EA for all $N > N_\alpha$.*

Proof. Assume that the seller and the type-t_0 bidder are risk neutral. Apply the results of Propositions 1 or 2 for Part (i), and of Proposition 3 for Part (ii). ■

We conclude this subsection with two more corollaries that are interesting on their own.

Corollary 2 *Suppose the bidders are risk neutral. Then the EA revenue is a mean-preserving spread of that of the EPA for all $\alpha \in (0, 1/2)$.*

Proof. By the revenue equivalence theorem, under bidder risk neutrality the expected revenue is the same in the EA and in the EPA. For risk neutral bidders A1-A6 hold trivially. Hence, by Theorem 3, the EPA revenue is preferred by all types of risk averse sellers. Hence the conclusion (e.g., Rothschild and Stiglitz, 1970). ■

This corollary generalizes a result of Goeree and Offerman (2004), in which they show that for uniformly distributed types the EPA revenue has a lower variance than that of the EA. Now, if we assume that the seller knows the utility functional form of u and w, and the distribution of the bidder types F, then the next corollary follows straightforwardly.

Corollary 3 *Under the assumptions of either Proposition 1 or Proposition 2, there exists an optimal* $\alpha^* \in (0, 1/2)$ *such that* $V_N(\alpha^*, p_0|EPA) \geq V_N(\alpha, p_0|EPA)$ *for all* $\alpha \in [0, 1/2]$.

Proof. Condition (i) of Theorem 3 implies that $V_N(1/2, p_0|\text{EPA}) = V_N(p_0|\text{EA}) = V_N(0, p_0|\text{EPA})$. Since $V_N(\alpha, p_0|\text{EPA})$ is positive for some α and is continuous, there exists an $\alpha^* \in (0, 1/2)$ that maximizes $V_N(\alpha, p_0|\text{EPA})$. ■

With sufficient knowledge about the distribution of the bidders' private information, the seller could also choose an optimal reserve price. In this case, interpreting p_0 as the optimal reserve price in the EA, all previous results in favor of the EPA extend trivially to the case in which the EPA reserve price is chosen optimally. One might also be interested in finding an incentive compatible, and individually rational, optimal auction that maximizes the seller's expected payoff. As demonstrated in the pioneering work of Matthews (1983) and Maskin and Riley (1984), however, such optimal auction under bidder risk aversion can be rather sophisticated, requiring strong assumptions for its existence.

4.2 Bidders' perspective

We now turn to the bidders' preferences for the auction forms. We show that prior to the auction all bidders whose marginal utilities upon winning decrease in their types will prefer the EPA to the EA. When type t is interpreted to be "wealth", this means that the premium auction format will be preferred by all risk averse bidders. This result is not immediate, however, because in the EA no bidder expects to end up with a loss, whereas in the EPA this can happen to a subset of bidder types.

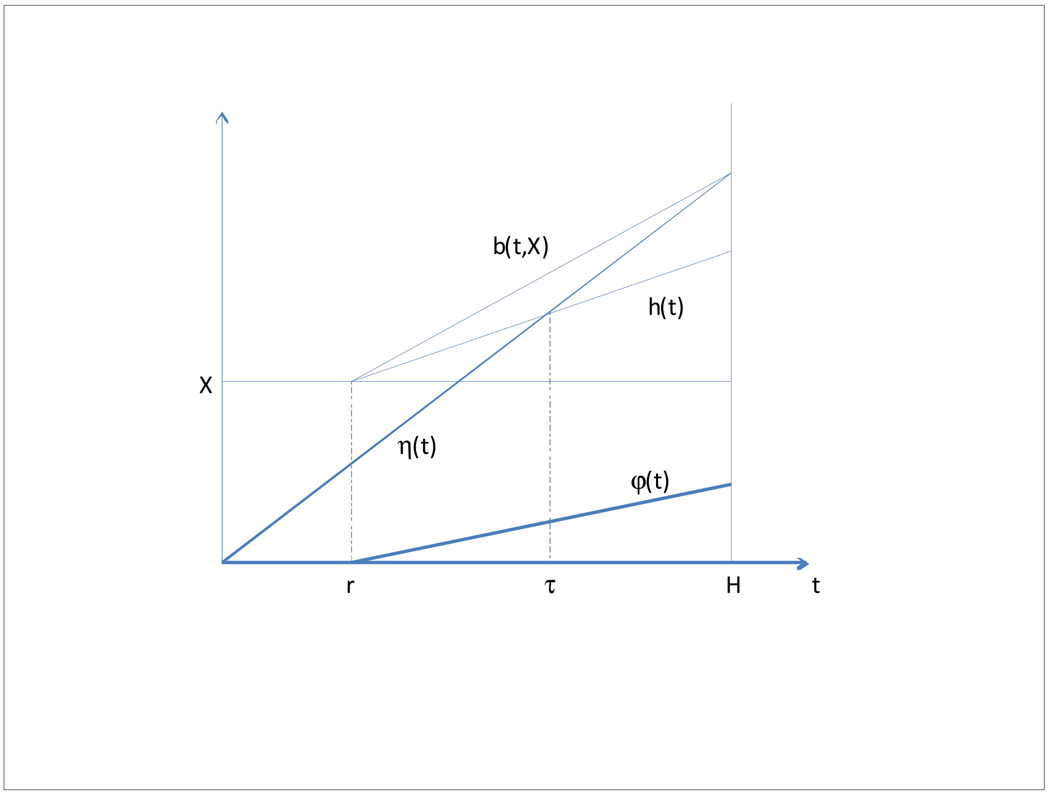

Figure 1: There is a threshold point τ at which the EA bid function $\eta(t)$ crosses the EPA effective payment function $h(t)$ from below.

To begin with, suppose there are $n > 2$ active bidders in the EPA under the reserve price p_0 (or else the EPA reduces to the EA). Since the first $n-2$ bidders who drop out are the same in either the EA or the EPA, who end up with the same status quo utility under either auction policy, it suffices to focus on the second stage EPA with any bottom price $X = \beta(r)$. To ease notation, we fix r and denote

$$\varphi(t) = \alpha(b(t,X) - X) \text{ and } h(t) = b(t,X) - \varphi(t)$$

Hence, $\varphi(t)$ is the premium and $h(t)$ is the *effective payment* by the winner when the EPA concludes at price $b(t,X)$. Figure 1 depicts the graphical relations among the functions b, η, φ, and h.

As implied by the boundary condition (7) and (9), the effective payment function $h(t)$ satisfies $h(H) < \eta(H)$. Since $h(r) > \eta(r)$, there is a crossing point τ at which $\eta(t) - h(t)$ switches the sign from negative to positive. To simplify the analysis, in what follows we assume that τ is unique.[21] This single crossing property implies that a finalist with type $t > \tau$ will be certain to attain a utility higher than $u(0,t)$ in either the winning or the losing situations. But if the finalist has type $t < \tau$, winning in the EPA could result in a utility level lower than $u(0,t)$. Figure 2 illustrates the two situations about a type-t finalist's ex post payoff as a function of the opponent's type t.

We first show the premium's effect on the bidders' expected payment.

Theorem 4 *For Cases 1'-4' suppose A1–A3, A5-A6 hold, and* $u_{11}(\cdot,t) < 0$ *for all* $t \in [0,H]$. *Suppose further for Case 3' that* $Q_1 = 0$. *Then the expected payment by any type of the bidders is lower in the EPA than in the EA.*

Proof. A type-t finalist's expected payment in the EA exceeds that in the EPA if

[21] Under additional assumptions (e.g., nondecreasing hazard rate of F) it can be shown that $h'(t) < \eta'(t)$ whenever $h(t) \leq \eta(t)$, which implies that the crossing point τ is unique.

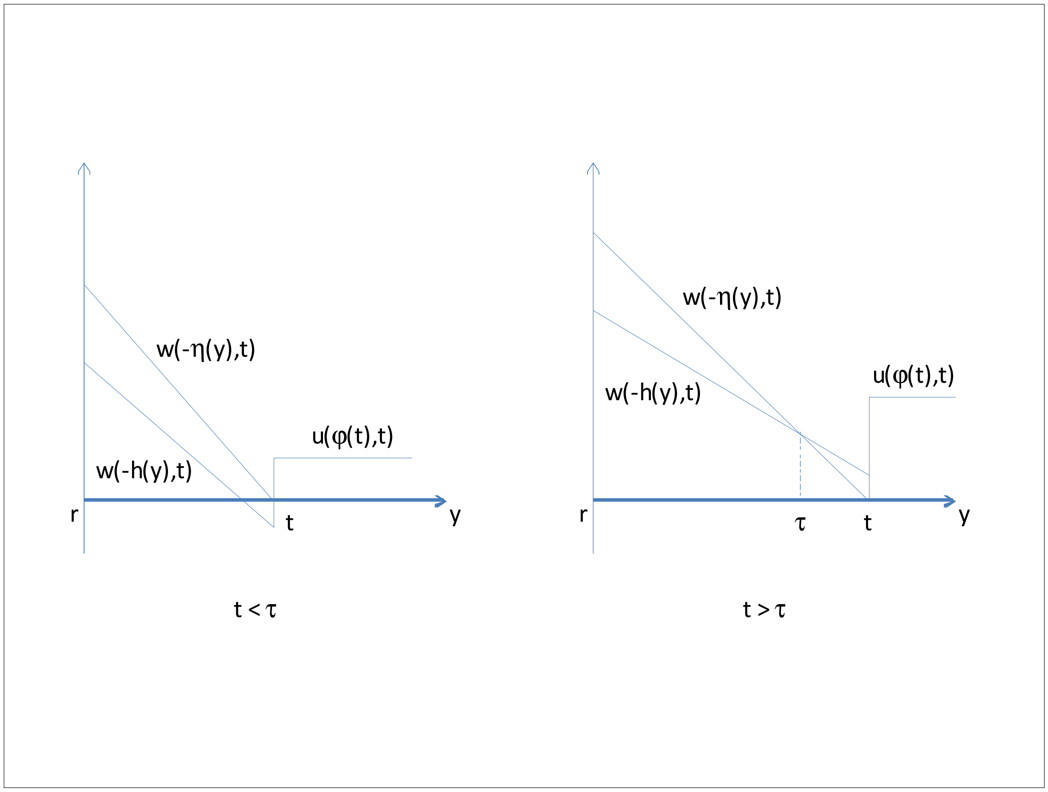

Figure 2: In situation $t < \tau$ the type-t finalist has the potential risk of losing money. This will happen when the opponent has a type $y < t$ that is sufficiently close to t. In situation $t \geq \tau$ the type-t finalist is ensured to earn a positive surplus.

and only if

$$A(t) \equiv \int_r^t (\eta(y) - h(y))\, dF(y) + (1 - F(t))\varphi(t) > 0$$

We have $A'(t) = (\eta(t) - b(t, X))\, f(t) + (1 - F(t)) a b_1(t, X)$. Substituting (8) gives

$$A'(t) = \left(\frac{u(\varphi, t) - w(-b + \varphi, t)}{u_1(\varphi, t)} - (b - \eta) \right) f(t)$$

Let $(\hat{u}, \hat{w})$ denote a risk neutral bidder population induced from some $\hat{U}(x, t)$ as in Cases 1'-4' such that $\hat{U}_{11} = 0$ and ($\psi' = 1$ for Case 2'). Because $u(\cdot, t)$ is risk averse, for Cases 1', 2', and 4' it can be readily verified that by A5, $A'(t) > 0$ for all $t \in (r, H]$. For Cases 3', $Q_1 = 0$ implies (by adding $b - \eta - \varphi$ to the first argument of Q)

$$\begin{aligned} A'(t) &= \left(\frac{u(b-\eta,t)-w(-\eta,t)}{u_1(b-\eta,t)}-(b-\eta)\right)f(t) \\ &= \left(\frac{u(b-\eta,t)-u(0,t)}{u_1(b-\eta,t)}-(b-\eta)\right)f(t)>0 \end{aligned}$$

where the inequality comes from u being risk averse. Since $A(r) = 0$, we have $A(t) > 0$ for all $t \in (r, H]$. ■

This Theorem generalizes our previous work (Hu, Offerman and Zou, 2011) by allowing the bidders to exhibit heterogeneous risk preferences and the auctioned object to carry ensuing risk.

We now compare the expected utilities of the two finalists. In the EA, when only two bidders remain, the expected utility of a type-t bidder equals

$$U(t|\text{EA}) = \frac{1}{1-F(r)}\int_r^t w(-\eta(y),t)dF(y) + \frac{1-F(t)}{1-F(r)}u(0,t)$$

The same bidder in the EPA has an expected utility equal to

$$U(t|\text{EPA}) = \frac{1}{1-F(r)}\int_r^t w(-h(y),t)dF(y) + \frac{1-F(t)}{1-F(r)}u(\varphi(t),t)$$

Therefore, in order to compare the bidders' preferences over the two auction forms it suffices to consider the sign of

$$\Delta(s,t) \equiv \int_r^s \left(w\left(-h(y),t\right)\right) - w(-\eta(y),t)\right) dF(y) + \left(1-F(s)\right)\left(u(\varphi(s),t)-u(0,t)\right)$$

and show that $\Delta(t,t) > 0$ for all $t \in (r, H]$.

Our last theorem shows the overall effect of the premium on the bidders' expected payoff.

Theorem 5 *For Cases 1'-4' suppose the assumptions of Theorem 4 hold and* $w_{12} < 0$*. Then, for all* $r \in [0, H)$ *and* $t \in (r, H]$, $U(t|EPA) > U(t|EA)$.

Proof. Fix any $r \in [0, H]$. First consider $t \in (r, \tau)$ so that $h(t) > \eta(t)$ (see Figures 1-2). Differentiating $\Delta(s,t)$ with respect to t yields

$$\Delta_2(s,t) = \int_r^s (w_2(-h(y),t)) - w_2(-\eta(y),t))\, dF(y) + (1-F(s))\,(u_2(\varphi(s),t) - u_2(0,t))$$

For the homogeneous-utility model, $\Delta_2(s,t) > 0$ by $w_{12} < 0$ and $h(s) > \eta(s)$. We now show that this is true also for the heterogeneous-utility model. Since $\varphi(r) = 0$, we have $\Delta_2(r,t) = 0$. Differentiating $\Delta_2(s,t)$ with respective to s gives

$$\begin{aligned}\Delta_{12}(s,t) &= (w_2(-h(s),t)) - w_2(-\eta(s),t))\, f(s) \\ &\quad + (1-F(s))u_{12}(\varphi(s),t)\alpha b_1(s,X) - (u_2(\varphi(s),t) - u_2(0,t))\, f(s)\end{aligned}$$

By $w_{12} < 0$ and $h(s) > \eta(s)$, the first term on the right side of this equation is positive. Substituting (8) for b_1, we obtain

$$\Delta_{12}(s,t) > \left(u_{12}(\varphi(s),t)\frac{u(\varphi(s),s) - w(-h(s),s)}{u_1(\varphi(s),s)} - (u_2(\varphi(s),t) - u_2(0,t))\right) f(s) \tag{21}$$

for all $s < t$. By Lemma 1, A5 implies

$$\begin{aligned}&\frac{\partial}{\partial t}\frac{u(x,t) - u(0,t)}{u_1(x,t)} \\ &= \left(u_2(x,t) - u_2(0,t) - u_{12}(x,t)\frac{u(x,t) - u(0,t)}{u_1(x,t)}\right)\frac{1}{u_1(x,t)} \leq 0\end{aligned} \tag{22}$$

Further, $h(s) > \eta(s)$ implies $w(-h(s),s) < w(-\eta(s),s) = u(0,s)$. Hence, by (22), we have

$$\begin{aligned}\frac{u(\varphi(s),s) - w(-h(s),s)}{u_1(\varphi(s),s)} &> \frac{u(\varphi(s),s) - u(0,s)}{u_1(\varphi(s),s)} \\ &\geq \frac{u(\varphi(s),t) - u(0,t)}{u_1(\varphi(s),t)} \text{ for all } s < t\end{aligned}$$

Hence, substituting $\varphi(s)$ for x in (22) and using the above inequality for (21), we obtain $\Delta_{12}(s,t) > 0$. It follows by integration that $\Delta_2(t,t) > 0$ for all $t \in (r, \tau]$. By the envelope theorem,

$$\frac{d}{dt}\Delta(t,t) = \Delta_2(t,t) > 0$$

Therefore, since $\Delta(r,r) = 0$ given $X = \beta(r)$, $\Delta(t,t) > 0$ for all $t \in (r,\tau]$.

Next consider $t > \tau$. Let $(\hat{u},\hat{w})$ be another bidder population induced from $\hat{U}(x,t)$ as in Cases 1'-4', and assume that all bidders in this population are risk neutral. Fix any $t \geq \tau$ and define

$$\pi(y) = \begin{cases} \hat{w}(-\eta(y),t) \text{ for } y \in [r,t] \\ \hat{u}(0,t) \qquad \text{for } y \in [t,H] \end{cases} \tag{23}$$

$$\rho(y) = \begin{cases} \hat{w}(-h(y),t) \text{ for } y \in [r,t] \\ \hat{u}(\varphi(t),t) \quad \text{for } y \in [t,H] \end{cases} \tag{24}$$

Further define μ as the induced probability distributions under the two auction formats:

$$\mu^{EA}(x) = \Pr(\pi(y) \leq x) \text{ and } \mu^{EPA}(x) = \Pr(\rho(y) \leq x) \tag{25}$$

By Theorem 4, the expected payment of each bidder is lower in the EPA than in the EA, hence

$$\begin{aligned} \hat{\Delta}(t,t) &\equiv \int_r^t \left(\hat{w}\left(-h(y),t\right)\right) - \hat{w}(-\eta(y),t))\, dF(y) + (1-F(s))\left(\hat{u}(\varphi(s),t) - \hat{u}(0,t)\right) \\ &= \int_r^H \left(\rho(y) - \pi(y)\right) dF(y) > 0 \end{aligned}$$

This is equivalent to

$$\int x d\mu^{EPA}(x) > \int x d\mu^{EA}(x) \tag{26}$$

From Figure 2 (event $t > \tau$), it can be seen that $\mu^{EPA}(x) - \mu^{EA}(x) < 0$ for low x and if $\mu^{EPA}(x) - \mu^{EA}(x) \geq 0$ then the sign remains positive for all $y \geq x$. This single crossing property together with (26) imply that $\mu^{EPA}(x)$ dominates $\mu^{EA}(x)$ in the sense of second-order stochastic dominance. Since the preference functions (u,w) are induced from $U(\cdot,t)$ that is risk averse for all t, $U(x,t) = G(\hat{U}(x,t),t)$ for some increasing (in x) and concave function G (e.g., Pratt, 1964). We therefore conclude that (e.g., Jewitt, 1987, Theorem 1), for Cases 1', 2', and 4',

$$\int G(x,t) d\mu^{EPA}(x) > \int G(x,t) d\mu^{EA}(x)$$

which is equivalent to $\Delta(t,t) > 0$. For Cases 3', because the object for sale involves ensuring risks, we need a more explicit analysis. In this case, $\hat{U}_{11}(x,t) = 0$ implies $\hat{U}(x,t) = C(t)x + \hat{U}(0,t)$ for some positive function $C(t)$. So,

$$\begin{aligned}
\hat{w}\left(x,t\right)) &= C(t)\int\left(v+x\right)dK(v|t) + \hat{u}(0,t) \\
&= C(t)\left(E\left(v|t\right)+x\right) + \hat{u}(0,t)
\end{aligned}$$

from which it follows that

$$\hat{w}\left(-h(y),t\right)) - \hat{w}(-\eta(y),t) = C(t)\left(\eta(y) - h(y)\right)$$

As the difference does not depend on $E\left(v|t\right)$, we can substitute any v for $E\left(v|t\right)$ in $\hat{w}$, re-define π and ρ as in (23)-(24), and $\mu^{EA}(x|v)$ and $\mu^{EPA}(x|v)$ as in (25) accordingly. This leads to

$$\int x d\mu^{EPA}(x|v) > \int x d\mu^{EA}(x|v) \quad \forall v$$

implying

$$\int G(x,t)d\mu^{EPA}(x|v) > \int G(x,t)d\mu^{EA}(x|v) \quad \forall v$$

Taking expectation gives

$$\int\int G(x,t)d\mu^{EPA}(x|v)dK(v|t) > \int\int G(x,t)d\mu^{EA}(x|v)dK(v|t)$$

which is equivalent to $\Delta(t,t) > 0$. The conclusions of the theorem are therefore established for all Cases 1'-4'. ■

Summarizing, this section has demonstrated a variety of circumstances in which the EPA Pareto dominates the EA at the interim, and therefore also at the ex ante stage of the auction game with incomplete information.

5 Conclusion

In this paper, we have presented an analysis of risk sharing effects in auctions with risk averse seller and bidders, while focusing on a simple single-object English auction environment in which the bidders' private information affects their mutual payoffs only through the premium rule.[22] Our study reveals that when both the seller and bidders are risk averse, the English auction is in general inefficient at the interim stage. This finding has significant normative and positive implications. Our analysis suggests that by simple modifications of the payment rule in the English auction, the auction designer can often make the auction more attractive to both the sellers and buyers when they are risk averse. Because of the preponderant evidence that the majority of individuals are risk averse and that people differ in their risk attitudes, the English premium auction studied in this paper could be of interest to designers of auctions in practice. On the positive side, the result of our study provides a plausible risk-sharing motive that helps explain why premium auctions have stood the test of time and remain a class of regularly adopted auctions in Europe.

An important related issue not considered in this paper is how rewarding premiums would affect the potential bidders' entry decisions (e.g., Levin and Smith, 1994; Smith and Levin, 1996; Bulow and Klemperer, 1996, 2009). We have assumed a fixed number of potential bidders and showed that risk averse bidders will unanimously prefer the EPA to the EA irrespective of the seller risk preference. Therefore, it is conceivable that when potential bidders make entry decisions based on their expected payoffs, and acquire information at some costs after entry, the EPA will be more conducive to entry than the EA. From the seller's viewpoint, this could increase revenue by more than an optimally structured auction does with fewer bidders (e.g., Bulow and Klemperer, 1996). So, even if the seller is risk neutral, the use of EPA could make sense for attracting more bidders. With endogenous entry, risk sharing between the seller and bidders in an EPA could improve ex post allocation

[22] Hu, Matthews and Zou (2012) study a similar English auction model with ensuing risk and heterogeneous bidders, allowing for a more general setting. Their focus is on the existence of ex post efficient equilibria and the effects of changing risk.

efficiency when more bidders are attracted to the trade, while an auction format that is more attractive for sellers could also encourage its actual usage.[23] In light of the unambiguous benefit of risk sharing among the players in the English premium auctions, one might also wish to know whether, and to what extent, similar improvement in risk sharing and Pareto efficiency can be found for other ex post efficient auctions when players are heterogeneous and risk averse. These will be interesting topics for future research.

Appendix

Proofs of the lemmas

Proof of Lemma 1. (i)$\Rightarrow$(ii) follows from Pratt (1964, Eqs. (21) and (22) for $y < 0$ and $y > 0$, respectively). (ii)$\Rightarrow$(iii) holds by replacing y in (3) by $y - \tilde{v}$, and taking expectation over $\tilde{v}$. (iii)$\Rightarrow$(i) holds by noting that if the weak [strong] form of (i) does not hold, then the strong [weak] form of (i) holds on some interval with u and $\hat{u}$ interchanged. Thus (iii) cannot hold true for all x, y, and $\tilde{v}$ (see Pratt, 1964; p. 129). ■

Proof of Lemma 2. Because $u(x, t)$ ($\equiv U(x)$) does not depend on t, $Q_3 < 0$ if and only if $w_2 > 0$. Hence, $U' > 0$ implies automatically A5 for Cases 1-3. This also holds for Case 4 by scaling U to be nonnegative on the relevant domain of definition, as assumed. Now suppose that U has nonincreasing risk aversion. Then, by Lemma 1, for Cases 1-3 $Q(x, y, t)$ is nonincreasing in x and therefore A6 holds. To see that

[23] For instance, Engelbrecht-Wiggans and Nonnenmacher (1999) documented how implementing a "seller friendlier" auction design in early nineteenth-century New York attracted more imports to the city and supported its subsequent economic growth. See van Bochove, Boerner and Quint (2012) for a historic account about the use of premium tactics in Europe.

it is also true with Case 4, note that in this case

$$\begin{aligned} Q(x,y,t) &= \frac{U(x)-(1+t)U(t+x-y)}{U'(x)} \\ &= (1+t)\frac{U(x)-U(t+x-y)}{U_1(x)} - t\frac{U(x)}{U'(x)} \end{aligned}$$

By Lemma 1 the first part is nonincreasing in x. By log-concavity, $\frac{U(x)}{U'(x)}$ is nondecreasing and therefore Q is nonincreasing in x. ■

Proof of Lemma 3. Given the assumptions on $U(x,t)$ and $K(v|t)$, the conditions A1-A4 hold trivially for the derived functions u and w. By Lemma 1, if $U(\cdot,t)$ increases in risk tolerance as t increases, then $Q(x,y,t)$ is a decreasing function of t, hence A5. Consider for example Case 3'. By Lemma 1, the assumption that $U(\cdot,t)$ is more risk averse than $U(\cdot,\hat{t})$ implies

$$\frac{U(x-(y-v),\hat{t})-U(x,\hat{t})}{U_1(x,\hat{t})} > \frac{U(x-(y-v),t)-U(x,t)}{U_1(x,t)} \qquad t<\hat{t}$$

Because $K(v|\hat{t})$ exhibits first-order stochastic dominance over $K(v|t)$, and because both sides of the above inequality increases in v, taking expectations maintains the inequality:

$$\frac{\int U(x-(y-v),\hat{t})dK(v|\hat{t})-U(x,\hat{t})}{U_1(x,\hat{t})} > \frac{\int U(x-(y-v),t)dK(v|t)-U(x,t)}{U_1(x,t)}$$

This shows $Q(x,y,t) > Q(x,y,\hat{t})$ and hence A5 is verified for Case 3'. The rest of the cases can be checked similarly.

As to A6, the condition is unaffected by substituting $U(x,t)$ for $U(x)$ in all Cases 1 to 4. Thus it holds for Cases 1'-4' by Lemma 2. ■

Proof of Lemma 4. The differential equation in (8) can be more succinctly written as

$$b_1(t,X) = \frac{1}{\alpha}Q(\alpha(b(t,X)-X), b(t,X), t)\frac{f(t)}{1-F(t)}$$

where Q is defined in (2). Because the right-hand side of (8) is continuously differentiable in b, t, and X, the solution $b(t,X)$ is continuously differentiable in t and X on its effective domain (e.g., Hale, 2009, Chapter 1, Theorem 3.3). Differentiating w.r.t. X gives

$$b_{12}(t,X) = -(1-b_2)Q_1 + \frac{1}{\alpha}Q_2 b_2 \frac{f(t)}{1-F(t)} \tag{27}$$

where

$$Q_1 = 1 - \frac{w_1}{u_1} - Q\frac{u_{11}}{u_1} \text{ and } Q_2 = \frac{w_1}{u_1} \tag{28}$$

By (9), substituting $b(H,X)$ for B in (7) gives

$$u(\alpha(b(H,X)-X),H) = w(-b(H,X)+\alpha(b(H,X)-X),H), \quad \forall X$$

Differentiating w.r.t. X yields, at $t = H$,

$$u_1 \times \alpha(b_2-1) = w_1 \times (-b_2 + \alpha(b_2-1)) \tag{29}$$

Part (i). Assume $Q_1 = 0$ and fix an arbitrary $X < B(H,H)$. Then from (28) we obtain $w_1 = u_1$ whenever $Q = 0$. Consequently, (29) implies $b_2(H,X) = 0$ as $w_1 > 0$. Now by (27), $b_2(t,X) \geq 0$ implies $b_{12}(t,X) \geq 0$. Hence $b_2 \leq 0$ for all $t \leq H$ such that $b(t,X) \geq X$ (see, e.g., Hu et al. 2011, Lemma 1). But this logic holds also for $-b_2$. Therefore, we must have $b_2(t,X) \equiv 0$ on the effective domain of b.

Part (ii). Assume $Q_1 < 0$. Then by (28), $Q = 0$ implies $w_1 > u_1$. Equation (29) now implies $w_1 b_2 = (w_1 - u_1)\,\alpha(b_2 - 1) < (w_1 - u_1)\,\alpha b_2$ and therefore $b_2(H,X) < 0$.

We first show that $b_2(t,X) < 1$ for all $t \in [r,H]$. This follows because by (27), $b_2(t,X) = 1$ implies $b_{12}(t,X) > 0$, which is impossible given $b_2(H,X) < 0$.

Now by (27), $b_2 = 0$ implies $b_{12}(t,X) > 0$. This implies that $b_2(t,X) < 0$ for all $t \leq H$ such that $b(t,X) \geq X$ (see, e.g., Hu et al. 2011, Lemma 1). ■

Proof of Lemma 5. By Theorems 1, substituting $X = \beta(t)$ in (8) yields $Q(0,\beta(t),t) > 0$ or equivalently, $u(0,t) > w(-\beta(t),t)$ for all $t \in [t_0,H)$. By (5) and $w_1 > 0$ we therefore have $\beta(t) > \eta(t)$ for all $t \in [t_0,H)$. ■

Proofs of the propositions

Proof of Proposition 1. By Theorem 3, it suffices to show that for all the cases considered, the function $\Phi(t)$ defined in (16) is positive. The conclusion has been established for Cases 1' in (20). For Cases 2'- 4', $Q_1 = 0$ and $w(-\eta(t), t) = u(0, t)$ imply $w(-\beta(t), t) = u(\eta(t) - \beta(t), t)$. So by (20) $\Phi(t) > 0$ for all $t > t_0$ under (19). ■

Proof of Proposition 2. By $Q_1 = 0$, the seller's break even condition $u(p_0, t_0) = w(0, t_0)$ holds iff $u(0, t_0) = w(-p_0, t_0)$. This implies that a sale occurs with an effective premium iff the pivotal bidder has a type $t > t_0$ (neglecting the zero-probability event of a tie at t_0). By the EA equilibrium $u(0, t) = w(-\eta(t), t)$, the assumption $Q_1 = 0$ also implies $u(\eta(t) - \beta(t), t) = w(-\beta(t), t)$. Hence,

$$\begin{aligned}
\frac{V(\beta(t)) - V(\eta(t))}{V'(\beta(t))} &= \frac{u(\beta(t), t_0) - u(\eta(t), t_0)}{u_1(\beta(t), t_0)} \quad \text{by assumption} \\
&= \frac{u(0, t_0) - u(\eta(t) - \beta(t), t_0)}{u_1(0, t_0)} \quad \text{by } Q_1 = 0 \\
&> \frac{u(0, t) - u(\eta(t) - \beta(t), t)}{u_1(0, t)} \quad \text{for all } t > t_0, \text{ by A5} \\
&= \frac{u(0, t) - w(-\beta(t), t)}{u_1(0, t)}
\end{aligned}$$

This shows that $\Phi(t) > 0$ so that by Theorem 3, $V_N(\alpha, p_0|\text{EPA}) > V_N(p_0|\text{EA})$. ■

Proof of Proposition 3. When the reserve price is zero, by A2 all bidders participate in the EA and hence in the EPA (Theorem 2). The inequality in (15) reduces to

$$V_N(\alpha, 0|\text{EPA}) - V_N(0|\text{EA}) \geq \int_0^H \Phi(t) V'(\beta(t))\, dF_{(2)}^N(t) \tag{30}$$

as can be seen by replacing 0 for t_0 in (15), given that all bidders are active under no reserve price. We have

$$\begin{aligned} &\int_0^H \Phi(t)V'(\beta(t))\,dF_{(2)}^N(t) \\ = &\int_0^{t_0} \Phi(t)V'(\beta(t))\,dF_{(2)}^N(t) + \int_{t_0}^H \Phi(t)V'(\beta(t))\,dF_{(2)}^N(t) \end{aligned}$$

where the last term is positive because $\Phi(t) > 0$ (by assumption that the seller is more risk averse than all types $t > t_0$). This term comes from the event $t_{(2)} \geq t_0$, and the probability of this event tends to 1 as N tends to infinity. Hence, because the term $\Phi(t)V'(\beta(t))$ is independent of N, for all $\alpha \in (0, 1/2)$ there exists an $N_\alpha > 2$ such that $\int_0^H \Phi(t)V'(\beta(t))\,dF_{(2)}^N(t) > 0$.

By A5, $\Phi(t)V'(\beta(t))$ has a single crossing property that if $\Phi(\hat{t})V'\left(\beta(\hat{t})\right) \geq 0$ for any $\hat{t}$ then $\Phi(t)V'(\beta(t)) > 0$ for all $t > \hat{t}$. Further notice that

$$\frac{f_{(2)}^{N+1}(t)}{f_{(2)}^N(t)} = \frac{(N+1)\,F(t)}{(N-1)}$$

is an increasing function of t. Thus (e.g., by Persico, 2000, Lemma 1)

$$\int_0^H \Phi(t)V'(\beta(t))\,dF_{(2)}^N(t) \geq 0$$

implies

$$\int_0^H \Phi(t)V'(\beta(t))\,dF_{(2)}^{N+1}(t) = \int_0^H \Phi(t)V'(\beta(t))\,\frac{f_{(2)}^{N+1}(t)}{f_{(2)}^N(t)}dF_{(2)}^N(t) \geq 0$$

The conclusion of the proposition thus holds true. ■

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