Implementing with veto players: simple mechanisms

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Abstract

The paper adapts a mechanism presented by Dagan, Serrano and Volij (1997) for bankruptucy problems to the context of TU veto balanced games.

1. Introduction

In 1997, Dagan, Serranoa and Volij present a simple tree games for bankruptcy problems. In the game a special player, the one with highest claim, has a special role. He is the proposer and the rest of the players answer this proposal. In the case of a negative answer the conflict is solved bilaterally, applying a normative solution concept to a special two-claimant bankruptcy problem. This paper studies two similar simple mechanisms in the context of coalitional games with veto players. In our models, a veto player is the proposer and similarly to the case of Dagan, Serrano and Volij, in the case of negative answer of some responder a bilateral resolution is formulated.

The paper shows that with the first mechanism the outcome of any Nash equilibrium should belong to a certain set. And those outcomes are not necessarily efficient.

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The second mechanism studies a more complex game where the veto player, the proposer, could make sequential proposals whenever there is a positive value to divide among the players. In this second model we connect a normative solution concept defined in the class of veto balanced games with the outcome of a special profile of strategies, a profile where the responders behave as myopic maximizers and the proposer is a rational player.

2. Preliminaries

A *cooperative n-person game in characteristic function form* is a pair (N, v), where N is a finite set of n elements and $v : 2^N \to \mathbb{R}$ is a real valued function on the family 2^N of all subsets of N with $v(\emptyset) = 0$. Elements of N are called *players* and the real valued function v the characteristic function of the game. Any subset S of the player set N is called a *coalition*. The number of players in a coalition S is denoted by $|S|$. Given a set of players N and a coalition $S \subset N$ we denote by S^c the set of players of N that are not in S. Generally we shall identify the game (N, v) by its characteristic function v.

A distribution among the players is represented by a real valued vector $x \in \mathbb{R}^N$ where x_i is the payoff assigned by x to player i. A distribution of an amount lower or equal to v is called a feasible distribution. We denote $\sum\limits_{i \in S} x_i$ by $x(S)$. A distribution satisfying $x(N) = v(N)$ is called an efficient allocation. An efficient allocation satisfying $x_i \geq v(i)$ for all $i \in N$ is called an *imputation* and the set of imputations is denoted by $I(v)$. The set of non negative feasible allocations is denoted by $D(N, v)$ and defined as follows

$$D(N, v) = \left\{ x \in \mathbb{R}^N : x(N) \leq v(N) \text{ and } x_i \geq 0 \text{ for all } i \in N \right\}.$$

The core of a game is the set of imputations that cannot be blocked by any coalition, i.e.

$$C(N, v) = \{ x \in I(v) : x(S) \geq v(S) \text{ for all } S \subset N \}.$$

A game with a nonempty core is called a balanced game. A game v is a *veto-rich game* if it has at least one veto player and the set of imputations is nonempty. *A player i is a veto player if* $v(S) = 0$ for all coalitions where player i is not present. A balanced game with at least one veto player is called a veto balanced game.

A solution ϕ on a class of games Γ_0 is a correspondence that associates with every game (N, v) in Γ_0 a set $\phi(N, v)$ in $\mathbb{R}^N$ such that $x(N) \leq v(N)$ for all $x \in \phi(N, v)$. If there is no confusion with the set of players we write (v) instead of (N, v). This solution is called it *efficient* if this inequality holds with equality. The solution is called it *single-valued* if for every game in the class the set contains a unique element.

Given a vector $x \in \mathbb{R}^N$ the *excess of a coalition S with respect to x* in a game v is defined as $e(S, x) := v(S) - x(S)$. Let $\theta(x)$ be the vector of all excesses at x arranged in non-increasing order of magnitude. The lexicographic order $\prec_L$ between two vectors x and y is defined by $x \prec_L y$ if there exists an index k such that $x_l = y_l$ for all $l < k$ and $\mathrm{x}_k < y_k$ and the weak lexicographic order $\preceq_L$by $x \preceq_L y$ if $x \prec_L y$ or $x = y$.

Schmeidler (1969) introduced the *nucleolus* of a game v, denoted by $\eta(v)$, as the unique imputation that lexicographically minimizes the vector of non increasingly ordered excesses over the set of imputations. In formula:

$$\{\eta(N, v)\} = \{x \in I(N, v) \,|\theta(x) \preceq_L \theta(y) \text{ for all } y \in I(N, v)\}.$$

For any game v with a nonempty imputation set, the nucleolus is a single-valued solution, is contained in the kernel and lies in the core provided that the core is nonempty.

In the class of veto balanced games the kernel, the prekernel and the nucleolus coincide (see Arin and Feltkamp (1997).

3. The game

Given a veto balanced game (N, v) and an order of players, we will define a tree game associated to the TU game and denoted by $G(N.v)$. The game has n stages and in each stage only one player is playing. In the first stage a veto player is playing and he anounces a proposal x^1 that belongs to the set of feasible and non negative allocations of the game (N, v). In the next stages the responders are playing, each one once at one stage. They have two actions. To accept or to reject. If a player, let say i, accepts the proposal x^{t-1} at stage t, he leaves the game with the payoff x_i^{t-1} and for the next stage the proposal x^t coincides with the proposal at $t-1$, that is x^{t-1}. If player i rejects the proposal then a two-person TU game is formed with the proposer and the player i. In this two-person game the value of the grand coalition is $x_1^{t-1} + x_i^{t-1}$ and the value of the singletons is obtained by applying the Davis-Maschler reduced game[1] given the game (N, v) and the allocation x^{t-1}. The player i will receive as payoff the result of some restricted standard solution applied in the two-person game. Once all the responders have played and consequently have received their payoffs the payoff of the veto player is also determined.

Formally, the resulting outcome of playing the game can be described by the following algorithm.

Input : a veto balanced game (N, v) with a veto player, the player 1, and an order in the set of the rest of the players (responders)
Output : a feasible and non negative distribution x.

1. Start with stage 1. The veto player makes a feasible and non negative proposal x^1 (not necessarily an imputation). Th superscript denotes at which stage the allocation is considered as the proposal in force.

2. In the next stage the first responder says yes or no to the proposal. If he says yes he receives the payoff x_2^1, leaves the game, and $x^2 = x^1$.

   If he says no he will receive the payoff

$$y_2 = \max\left\{0, 1/2(x_1^1 + x_2^1 - v_{x^1}(\{1\}))\right\} \text{ where}$$
$$v_{x^1}(\{1\}) = \max_{1\in S\subseteq N\setminus\{2\}} \left\{v(S) - x^1(S\setminus\{1\})\right\}$$

   Now, $x^2 = \begin{cases} x_1^1 + x_2^1 - y_2 & \text{for player 1} \\ y_2 & \text{for player 2} \\ x_i^1 & \text{if } i \neq 1, 2. \end{cases}$

[1] Let (N, v) be a game, $T \subset N$, and consider $T \neq N, \emptyset$ and a feasible allocation x. Then the *Davis-Maschler reduced game* with respect to $N \setminus T$ and x is the game $(N \setminus T, v_x)$ where

$$v_\phi^{N\setminus T}(S) := \begin{cases} 0 \quad \text{if } S = \emptyset \\ v(N) - \sum\limits_{i\in T} x_i \quad \textit{if} \;\; S = N \setminus T \\ \max\limits_{Q\subset T}\left\{v(S\cup Q) - \sum\limits_{i\in Q} x_i\right\} \qquad \text{for all } S \subset N \setminus T. \end{cases}$$

We also denote the game $(N \setminus T, v_x)$ by $v_x^{N\setminus T}$. Note that we define a modified Davis-Maschler reduced game where the value of the grand coalition of the reduced game is obtained in a different way. In our case, $.v(N\setminus T) = \sum\limits_{i\in N\setminus T} x_i$.

3. Let the stage t where the t responder plays, given the allocation x^{t-1}. If he says yes he receives the payoff x_t^{t-1}, leaves the game, and $x^t = x^{t-1}$.

   If he says no he will receive the payoff

$$y_t = \max\left\{0, 1/2(x_1^{t-1} + x_t^{t-1} - v_{x^{t-1}}(\{1\}))\right\} \text{ where}$$

$$v_{x^{t-1}}(\{1\})) = \max_{1\in S\subseteq N\setminus\{t\}} \left\{v(S) - x^{t-1}(S\setminus\{1\})\right\}$$

   Now, $x^t = \begin{cases} x_1^{t-1} + x_2^{t-1} - y_t & \text{for player } 1 \\ y_t & \text{for player } t \\ x_i^{t-1} & \text{if } i \neq 1, 2 \end{cases}$

4. The game ends when the stage n is played and we define $x^n(N, v)$ as the vector with coordinates $\left(x_j^n\right)_{j\in N}$.

In this game we assume that the conflict between the proposer and a responder is solved bilaterally. In the case of conflict, the players face a two-person TU game that shows the strength of the players given the fact that the rest of the responders do not play. Once the game is formed the allocation proposed for the game is a normative proposal, a kind of restricted standard solution[2]. It is restricted because non negative payoffs are not allowed. If the formed two-person game is balanced, the solution will be the standard solution that coincides with the prekernel and the nucleolus.

4. The Nash outcomes

The main question we try to solve is which outcomes we can expect from the equilibria of the game (we call a Nash outcome the vector of payoffs associates to a Nash equilibrium). One may think that the prekernel of the game is a good candidate to be an outcome of the equilibria since in case of conflict, in mant

[2] Given a two-person game $(\{1,2\}, v)$ we called standard solution the following vector: $(v(1)+d, v(2)+d)$ where $d = \frac{v(1,2)-v(1)-v(2)}{2}$. In our model we apply the solution whenever the payoffs are non negative. The main results of the paper do not change if we use the standard solution instead of the restricted standard solution as the concept with which we solve the bilateral conflict. Since our main idea is to discuss simple mechanisms we think is more credible to assume that no player will accept a negative payoff, a payoff lower than his individual worth.

cases, the players solve the situation applying the prekernel of a game obtained with the Davis-Maschler reduction[3].

The first example shows that the nucleolus, in general, is not the outcome of equilibrium of the game $G(N, v)$.

Example 4.1. *Let* $N = \{1, 2, 3, 4, 5, 6\}$ *a set of players and consider the following 6-person veto balanced game* (N, v) *where*

$$v(S) = \begin{cases} 1 & \text{if } |S| > 2 \text{ and } 1 \in S \text{ and } S \neq N \\ 3 & \text{if } S = N \\ 0 & \text{otherwise.} \end{cases}$$

Computing the nucleolus[4] of this game we see that all the players receive the same payoff. It can be immediately checked that if the proposer starts with the following proposal $x^1 = (1, 1, 1, 0, 0, 0)$ after the optimal[5] answer of the rest of the players the final outcome it will be the vector x^1. Therefore it is clear that in this case the outcome of an equilibrium cannot be the nucleolus.

The main theorem of this section gives necessary and sufficient conditions to identify all the Nash outcomes of the game. We need some definitions and lemmas before introducing this main theorem.

Given a game (N, v) and a feasible allocation x we define the complaint of the player i against the player j as follows:

$$f_{ij}(x) = \min_{i \in S \subseteq N \setminus \{j\}} \{x(S) - v(S)\}.$$

The set of allocations bilaterally balanced for player i is

$$F_i(N, v) = \{x \in D(N, v) : f_{ji}(x) \geq f_{ij}(x) \text{ for all } j \neq i\}$$

while the set of optimal elements for player i in the set $F_i(N, v)$ is defined as follows:

[3] See Maschler (1992). The prekernel satisfies the Davis-Maschler reduced game property and coincides with the standard solution in the case of two-person games.

[4] Arin and Feltkamp (1997) present an algorithm for computing the nucleolus in the class of TU games with veto players.

[5] If the responders are not playing optimally it is not true that with this proposal the final payoff of the veto player will be at least 1. But that is sure if the initial proposal is the vector $(1, 0, 0, 0, 0, 0)$.

$$B_i(N,v) = \arg\max_{x \in F_i(N,v)} x_i.$$

Note that since $F_i(N,v)$ is a nonempty (it contains the prekernel[6]) compact set the set $B_i(N,v)$ is nonempty.

Lemma 4.2. *Let be (N,v) a veto balanced TU game and let be $G(N,v)$ its associated tree game. Given a proposal at stage t the responder playing optimally at this stage, say player i, will reject the proposal x^{t-1} if $f_{i1}(x^{t-1}) < f_{1i}(x^{t-1})$ and will accept the proposal if $f_{i1}(x^{t-1}) > f_{1i}(x^{t-1})$. If $f_{i1}(x^{t-1}) = f_{1i}(x^{t-1})$ the player is indifferent between accepting or rejecting.*

Proof. The responder playing at stage t should compare the amount y_i resulting after rejection with the amount x_i^{t-1} resulting after accepting. Note that $x_i^{t-1} = f_{i1}(x^{t-1})$ and $v_{x^{t-1}}(\{1\}) = -f_{1i}(x^{t-1}) + x_1^{t-1}$. Therefore after rejection, it holds that $y_i = 1/2(f_{1i}(x^{t-1}) + f_{i1}(x^{t-1})$.

Therefore $y_i > x_i^{t-1}$ if and only if $f_{i1}(x^{t-1}) < f_{1i}(x^{t-1})$. ■

Note that if a player i rejects optimally the proposal x^{t-1} at stage t it holds that $f_{i1}(x^t) = f_{1i}(x^t)$. This is so because

$f_{1i}(x^t) = f_{1i}(x^{t-1}) - (x_1^{t-1} - x_1^t)$ and

$x_1^t = x_1^{t-1} + f_{i1}(x^{t-1}) - y_i$.

Combining both equalities and knowing that $y_i = 1/2(f_{1i}(x^{t-1}) + f_{i1}(x^{t.1})$ we get $f_{i1}(x^t) = f_{1i}(x^t)$.

Lemma 4.3. *Let be (N,v) a veto balanced TU game and let be $G(N,v)$ its associated tree game. Given any proposal x^1, and if the responders play best response strategies, the final outcome of the game will be an element of $F_1(N,v)$. That is, $x^n \in F_1(N,v)$.*

Proof. Let i be a responder playing his best response at stage t. If he says yes we have that $f_{i1}(x^{t-1}) \geq f_{1i}(x^{t-1})$ and if he rejects we will have that $f_{i1}(x^t) = f_{1i}(x^t)$. It is also clear that if all responders play optimally then $x_1^t \geq x_1^{t+1}$ for all $t \in \{1, ..., n-1\}$. Note also that in each stage if any, there is only one bilateral transfer from the proposer to a responder. Let be l the responder playing at stage

[6] If we denote by PK the prekernel then $PK(N,v) = \bigcap_{i \in N} (F_i(N,v) \cap I(N,v))$.

$t+1$. If player l accepts it is clear that $f_{1i}(x^t) = f_{1i}(x^{t+1})$. If player l rejects then either $f_{1i}(x^t) = f_{1i}(x^{t+1})$ or $f_{1i}(x^t) > f_{1i}(x^{t+1})$ depending on which players contain the coalition that the proposer is using to complain against the responder i, .Therefore $f_{1i}(x^t) \geq f_{1i}(x^{t+1})$ for all $t \in \{1, ..., n-1\}$ and for all $i \neq 1$. ■

Remark 1. *Note that as a consequence of the lemma if the initial proposal belongs to $F_1(N, v)$ then, assuming optimal behavior of the responders, the final proposal will coincide with the initial proposal. That means that the proposer can guarantee a payoff for him just proposing an element of $B_1(N, v)$.*

The following theorem is a result of this implication.

Theorem 4.4. *Let be (N, v) a veto balanced TU game and let be $G(N, v)$ its associated tree game. Let z be a feasible and non negative allocation. Then z is a Nash outcome if and only if $z \in B_1(N, v)$.*

Proof. Let $z \in B_1(N, v)$ and consider the following profile of strategies: z is proposed by the proposer and the responders respond at any proposal rejecting if and only if after rejection they increase their payoff. If not they accept. It is immediate that this profile is a Nash equilibrium for which the final payoff vector is z.

Let z be a Nash outcome. By Lemma 4,3 $z \in F_1(N, v)$. Let $k = \max_{x \in F_1(N,v)} x_1$. By definition $z_1 \leq k$. By Remark 1 $z_1 \geq k$. Therefore $z_1 = k$ and consequently $z \in B_1(N, v)$. ■

Remark 2. *The result of Theorem 4.4 is independent of the order of the responders. It is valid for any order.*

Analyzing again Example 4.1 we can check that the set $B_1(N, v)$ could contain more than one element. To prove that first of all we will prove that if z belongs to $B_1(N, v)$ then $z_1 = 1$.

Suppose $z = (z_1, z_2, z_3, z_4, z_5, z_6) \in B_1(N, v) \subset F_1(N, v)$ and $z_1 > 1$. Therefore $z_i = f_{i1}(z) \geq f_{1i}(z)$ for all $i \neq 1$. Let be i the non veto player with lowest payoff according to z. If $z_1 > 1$ it is clear that $f_{1i}(z) > 2z_i \geq z_i$. Therefore it is not true that $z \in F_1(N, v)$.

It can be checked that any vector x, such that $x_1 = 1$ and at least three responders receive 0 will be an element of $B_1(N, v)$.

Note that, if the responders are playing optimally, any proposal of the proposer ending in an outcome of $B_1(N,v)$ will be a best strategy for the proposer.

Because of those reasons we call the elements of the set $B_1(N,v)$ outcomes of equilibrium of the game $G(N,v)$.

We have seen that the elements of $B_1(N,v)$ are not necessarily imputations. In some cases, the set $B_1(N,v)$ does not contain any efficient allocation.

Example 4.5. *Let $N = \{1,2,3,4,5\}$ a set of players and consider the following 5-person veto balanced games (N,v) and (N,w) where*

$$v(S) = \begin{cases} 8 & \begin{array}{c} \text{if } |S| > 3 \\ \text{and } 1 \in S \end{array} \\ 0 & \text{otherwise} \end{cases} \quad \text{and } w(S) = \begin{cases} 8 & \begin{array}{c} \text{if } |S| > 3,\ 1 \in S \\ \text{and } S \neq N \end{array} \\ 12 & \text{if } S = N \\ 0 & \text{otherwise.} \end{cases}$$

It is clear that for the game $G(N,v)$ the proposal $x^1 = (8,0,0,0,0)$ is the optimal strategy for the proposer and the final outcome will be x^1. The result does not depend on the strategies of the responders.

For the game $G(N,w)$ it is still true that the proposal x^1 will end in itself independently of the strategies of the responders. Therefore, any equilibrium should generate an outcome in which the final payoff of the proposer is at least 8. But that is not possible of the proposer is forced to do efficient proposals. The reason is that any imputation z in which z_1 is equal or higher than 8 is not an element of $F_1(N,v)$ as the following argument shows.

Suppose $z = (z_1, z_2, z_3, z_4, z_5)$ is an efficient outcome of some equilibrium in the game $G(N,w)$ and that $z_1 \geq 8$. By lemma 4.3 $z \in F_1(N,v)$. Therefore $z_i = f_{i1}(z) \geq f_{1i}(z)$ for all $i \neq 1$. But, if $z_1 \geq 8$ it is true that $f_{1i}(z) \geq \sum\limits_{l \neq 1,i} z_l > z_i$ for at least the responder with lowest payoff. Therefore it is not true that $z \in F_1(N,v)$.

Next example shows that, in some cases, the order of the responders influences the outcome of equilibrium[7].

Example 4.6. *Let $N = \{1,2,3,4\}$ a set of players and consider the following 4-person veto balanced game (N,v) where*

[7] The example does not contradict Remark 2. The set of Nash outcomes of a game $G(N,v)$ is independent of the order of the responders.

$$v(S) = \begin{cases} 1 & \text{if } |S| > 1,\ 1 \in S \\ & \text{and } S \neq N \\ 1.5 & \text{if } S = N \\ 0 & \text{otherwise.} \end{cases}$$

Suppose the following proposal: $(1.5, 0, 0, 0)$. Given this proposal it can be checked that the final outcome of the game if the players play optimally will be the following: $x_1 = 1$, the player answering the first gets 0.25 and the last two responders obtain 0.125 each one. Therefore this outcome depends on the order of the responders.

Moreover the final outcome, $(1, 0.25, 0.125, 0.125)$, an element of the set $B_1(N, v)$, a ash outcome. To see that we need to prove that if $z \in B_1(N, v)$ then $z_1 = 1$. Suppose on the contrary that there is z such that $z \in B_1(N, v)$ and $z = (1+\varepsilon, z_2, z_3, z_4)$ where $\varepsilon > 0$. Without loss of generality let z_4 be the lowest payoff and z_3 the second lowest payoff.

Since $z \in F_1(N, v)$ the following inequalities hold:
$f_{31}(z) = z_3 \geq f_{13}(z) = \varepsilon + z_4$,
$f_{41}(z) = z_4 \geq f_{12}(z) = \varepsilon + z_3$,
and combining both inequalities we have the following contradiction
$z_4 \geq \varepsilon + z_3 \geq \varepsilon + \varepsilon + z_4$, since $\varepsilon > 0$.

The example shows that the set $B_1(N, v)$ does not satisfy equal treatment property[8], that is, equal players do not receive equal payoff. It is clear, that the set always contain elements where equal players are treated equally since it is always possible to decrease the payoff of the player with highest payoff until the payoffs are equalized.

Those results do not change if the proposer is a non veto player. That is, given a TU veto balanced game and its associated tree game $G(N, v)$ where the proposer is the player i, no necessarily a veto player, the outcome of any equilibrium of the game will be an element of $B_i(N, v)$. And any element of $B_i(N, v)$ is the result of at least one equilibrium. The proofs should be modified slightly taking into account the following equalities:

$$v_{x^{t-1}}(\{k\}) = -f_{kl}(x^{t-1}) + x_k^{t-1} \text{ and } v_{x^{t-1}}(\{l\}) = -f_{lk}(x^{t-1}) + x_l^{t-1}.$$

[8] A solution ϕ satisfies **equal treatment property** if for each (N, v) in Γ_0 and for every $x \in \phi(N, v)$ interchangeable players i, j are treated equally, i.e., $x_i = x_j$. Here, i and j are interchangeable if $v(S \cup i) = v(S \cup j)$ for all $S \subseteq N \backslash \{i, j\}$.

And consequently

$$\begin{aligned} y &= 1/2(x_l^{t-1} + x_k^{t-1} - v_{x^{t-1}}(\{l\}) - v_{x^{t-1}}(\{k\})) = \\ &= 1/2(f_{kl}(x^{t-1}) + f_{lk}(x^{t-1})). \end{aligned}$$

In fact, similar results can be obtained if we take a TU game without veto players as the game with which we formulate the game $G(N, v)$. In this case it is not clear how to determine the proposer.

5. A new game: sequential proposals

When the game is played and the final outcome is not efficient (Example 4.5) a new proposal can be made. With this idea in mind we model a new game where the proposer can make sequential proposals, and each proposal is answered by the responders as in the previous model. Again, given a veto balanced game with a proposer and an order in the set of responders we will construct a tree game, denoted by $G^2(N, v)$.

Formally, the resulting outcome of playing the game can be described by the following algorithm.

Input : a veto balanced game (N, v) with a veto player, the player 1, and an order in the set of the rest of the players (responders)
Output : a feasible and non negative distribution x.

1. Start with stage 1. Given a veto balanced TU game (N, v) and an order in the set of responders (the order is fixed for all the stages) we play the game $G(N, v)$ and define the veto balanced TU game (N, v^{2,x_1}) where $v^{2,x_1}(S) = \max\{0, v(S) - x^1(S)\}$ and x_1 is the final outcome obtained in the first stage. Then go to the next step. The superscripts in the characteristic function denote at which stage and after which outcome the game is considered as the game in force. If no confusion we write v^2 instead of v^{2,x_1}.

2. Let be the stage t (t<n+1) and the TU game $(N, v^{t,x_{t-1}})$. We play the game $G(N, v^t)$ and define the veto balanced TU game (N, v^{t+1,x_t}) where $v^{t+1}(S) = \max\{0, v^t(S) - x^t(S)\}$ and x_t is the final outcome obtained in the previous stage. Then go to the next step.

3. The game ends after stage n. (If at some stage before n the proposer makes an efficient proposal (efficient according to the TU game underlying at this stage) the game is trivial for the rest of the stages).

4. The outcome after playing the game $G^2(N, v)$ is the sum of the outcomes generated at each stage.

We introduce now a solution concept defined on the class of veto balanced games and denoted by ϕ. This solution concept will be related to the tree game we have presented.

Let $d_i = \max_{S \subseteq N \setminus \{i\}} v(S)$ for all i. Then $d_1 = 0$. Let $d_{n+1} = v(N)$ and rename players according to the nondecreasung order of those values. That is, player 2 is the player with the lowest value and so on. Define the solution ϕ as follows:

$$\phi_l = \sum_{i=l}^{n} \frac{d_{i+1} - d_i}{i} \text{ for all } l \in \{1, ..., n\}.$$

It is clear that in the class of veto balanced games the solution ϕ satisfies some well-known properties as nonemptiness, efficiency, anonymity[9] and equal treatment property among others. It also satisfies aggregate monotonicity[10]. In the paper of Dagan, Serrano and Volij it is discussed that non monotonic solutions, like the nucleolus[11] are not implementable via those simple mechanisms. This conjecture is true in the first model. In this second model we will see that the monotonic solution we have introduced can be considered as the unique outcome of a special equilibrium, a equilibrium where all the responders play myopically.

Lemma 5.1. *Let be (N, v) a veto balanced TU game and $G^2(N, v)$ its associated tree game. Let z be a Nash outcome of the game $G^2(N, v)$. Then* $z_1 \geq \phi_1 = \sum_{i=1}^{n} \frac{d_{i+1} - d_i}{i}$.

Proof. The result is based on the fact that the proposer has a strategy with which he will guarantee the payoff $\sum_{i=1}^{n} \frac{d_{i+1} - d_i}{i}$ independently of the strategies of the rest of the players.

[9] A solution ϕ satisfies **anonymity** if for each (N, v) in Γ_0 and each bijective mapping $\tau : N \longrightarrow N$ such that $(N, \tau v)$ in Γ_0 it holds that $\phi(N, \tau v) = \tau(\phi(N, v))$ (where $\tau v(\tau T) = v(T)$, $\tau x_{\tau(j)} = x_j$ $(x \in R^N, j \in N, T \subseteq N)$). In this case v and τv are equivalent games.

[10] Let ϕ be a single-valued solution on a class of games Γ_0. We say that solution ϕ satisfies:

Aggregate-monotonicity property (Meggido, 1974): for all $v, w \in \Gamma_0$, if for all $S \neq N$, $v(S) = w(S)$ and $v(N) < w(N)$, then for all $i \in N$, $\phi_i(v) \leq \phi_i(w)$.

[11] See Arin and Feltkamp (2001) for a study of monotonicity properties of the nucleolus in the class of veto balanced games.

The strategy is the following. At each stage t, $(t \in \{1, ..., n\})$ the proposal will be

$$x_l^t = \begin{cases} \frac{d_{t+1}-d_t}{|S|} & \text{for all } l \in S \text{ and } S = \{l;\ d_l \leq dt\} \\ 0 & \text{otherwise.} \end{cases}$$

It can be checked immediately that in each stage the proposed allocation will be the final allocation independently of the answers of the responders and independently of the order of those answers. Therefore this strategy of the proposer determines the total payoff of all the players, that is, the final outcome of the game $G^2(N, v)$. This final outcome coincides with the solution ϕ. ■

In the following we study the strategies of the responders. A special strategy is the one in which the responders behave optimally at each subgame independently of the influence of such behavior in the following subgames. If all the responders behave as myopic maximizers and the proposer is playing optimally the resulting outcome is unique. The next result solves the following question: Suppose all the responders answer optimally but myopically in each subgame and that the proposer is playing optimally the game, which outcomes we can expect? We call myopic best response strategies (MBRE) such profile of strategies.

Lemma 5.2. *Let be (N, v) a veto balanced TU game and $G^2(N, v)$ its associated tree game. Let i be a responder and w be an outcome resulting of some MBRE of the game $G^2(N, v)$. Then* $w_i \geq w_1 - \sum\limits_{l=2}^{i} \frac{d_l - d_{l-1}}{l-1}$ *for all* $i \in \{2, ..., n\}$ *.*

Proof. By induction on the players index. We will prove that $w_2 \geq w_1 - d_2$. Suppose on the contrary that $w_2 < w_1 - d_2$. That means that there exists a subgame after which the difference of accumulated payoffs between player i and player 2 is higher than d_2. Let be $G(N, v^k)$ this subgame in which we have that $y_1(t-1) + x_1^t - (y_2(t-1) + x_2^t) > d_2$ where x^t is the proposal faced by player 2 in the subgame and $y_i(t-1)$ is the total payoff received by the player i in the previous subgames. If player 2 rejects the proposal x^t we have the following two person game :

$v_{x^t}(\{1\}) \leq d_2 - y_1(t-1)$

$v_{x^t}(\{2\}) = 0$ and

$\text{v}_{x^t}(\{1, 2\} = x_1^t + x_2^t$.

Computing the solution for this game we see that

$$y_2 \geq \frac{1}{2}(x_1^t + x_2^t - (d_2 - y_1(t-1))) > \frac{1}{2}((y_2(t-1) + x_2^t) + x_2^t)) > x_2^t.$$

Hence accepting is not myopically optimal. After rejecting,

$$y_1 + y_1(t-1)) - y_2 \leq d_2.$$

If the rest of the responders behave myopically optimizing then y_1 can only decrease. Hence, the inequality holds true.

Therefore player 2 can always guarantee that the difference between the final payoffs is no higher than d_2. Even more, player 2 determines when the difference between the proposer and the player 2 is going to be limited by d_2. If player 2 does no complain the stage where the difference is violated he can not guarantee anymore that this difference should be satisfied. That means that if at some stage l of the game the accumulated payoff of the proposer is higher than d_2 then at this stage the accumulated payoff of the player 2 is equal or higher than $z_1 - d_2$, where z_1 is the accumulated payoff of the proposer at this stage. And this result holds indepently of the order in the set of responders, for any order.

Suppose that the lemma is true for players $2, 3, ..., k-1$. We need to prove that $w_k \geq w_1 - \sum_{l=2}^{k} \frac{d_l - d_{l-1}}{l-1}$. Suppose on the contrary that $w_k < w_1 - \sum_{l=2}^{k} \frac{d_l - d_{l-1}}{l-1}$. That means that there exists a subgame after which the difference of accumulated payoffs between player i and player k is higher than $\sum_{l=2}^{k} \frac{d_l - d_{l-1}}{l-1}$. Let be $G(N, v^k)$ this subgame in which we have that $y_1(t-1) + x_1^t - (y_k(t-1) + x_k^t) > \sum_{l=2}^{k} \frac{d_l - d_{l-1}}{l-1}$ where x^t is the proposal faced by player k in the subgame. If player k rejects the proposal x^t we have the following two person game :

$v_{x^t}(\{1\}) \leq \max_{p \in \{2,...,k\}} \left\{ d_p - \sum_{l=2}^{p-1} q_l - y_1(t-1) \right\}$

$v_{x^t}(\{2\}) = 0$ and

$v_{x^t}(\{1, 2\} = x_1^t + x_k^t$,

where we denote by q_l the accumulated payoff obtained by player l up in the stage k when player k is playing. The proposer has different coalitions with which he can get his value in the bilateral game. The highest value that a coalition could have without player k is d_k. It is also immediate that the coalitions with value d_k should contain all the responders preceding[12] player k. Therefore the payoffs of all this players should be taken into account. The same occurs if the proposer decides to use a coalition with a value of d_{k-1}. In this case all the players preceding player $k-1$ belong to such coalitions. Therefore we get $v_{x^t}(\{1\}) \leq \max\limits_{p\in\{2,\ldots,k\}} \left\{ d_p - \sum\limits_{l=2}^{p-1} q_l - y_1(t-1) \right\}$.

Note that since all the accumulated payoffs are non negative if $z_k < z_1 - \sum\limits_{l=2}^{k} \frac{d_l - d_{l-1}}{l-1})$ then $z_1 > \sum\limits_{l=2}^{k} \frac{d_l-d_{l-1}}{l-1}$. Note also that a player m before the player k has an accumulated payoff at the end of stage p that equals $z_m \geq t_m$, since player m is myopic maximizer. The difference of accumulated payoffs between the proposer and the player m preceding player k has been fixed or is going to be fixed at the end of the stage. And it holds $z_m \geq \sum\limits_{l=m}^{k} \frac{d_l-d_{l-1}}{l-1}$.

Computing the difference between the accumulated payoffs of the proposer and player k at the end of stage k after rejection of the proposal x^t we get

$$z_1 - z_k \leq y_1(t-1) + x_1^t + x_k^t - (x_1^t + x_k^t - v_{x^t}(\{1\}) \leq$$
$$\max_{p\in\{2,\ldots,k\}} \left\{ d_p - \sum_{l=2}^{p-1} q_l \right\} \leq$$
$$\max_{p\in\{2,\ldots,k\}} \left\{ d_p - \sum_{l=2}^{p-1} z_l \right\} \leq \max_{p\in\{2,\ldots,k\}} \left\{ d_p - \sum_{l=2}^{p-1} \frac{l-1}{l}(d_{l+1} - d_l) \right\} = \sum_{l=2}^{k} \frac{d_l-d_{l-1}}{l-1}.$$

The last inequality is a direct consequence of the fact that $z_p \geq \sum\limits_{l=p}^{k} \frac{d_l-d_{l-1}}{l-1}$ for all responder p preceding the responder k. Therefore even if the difference is not established immediately after the decision of rejecting or accepting, this difference is going to appear after the response of all the players in the subgame k.

Note that the result holds for any order of the responders. ■

Combining lemma 5.1 and lemma 5.2 we get the main result.

Theorem 5.3. *Let be (N, v) a veto balanced TU game and $G^2(N, v)$ a tree game. Let z be an outcome resulting from a MBRE of the game $G^2(N, v)$. Then* $z = \phi$.

Proof. By Lemma 5.1 $z_1 \geq \phi_1$. And by Lemma 5.3 for all $l \in \{2, ..., n\}$ it holds $z_l \geq z_1 - \sum\limits_{l=2}^{i} \frac{d_l-d_{l-1}}{l-1} \geq \phi_1 - \sum\limits_{l=2}^{i} \frac{d_l-d_{l-1}}{l-1} = \sum\limits_{l=2}^{n+1} \frac{d_l-d_{l-1}}{l-1} - \sum\limits_{l=2}^{i} \frac{d_l-d_{l-1}}{l-1} = \phi_l$.

[12] By preceding we mean players that have a value d_i lower than d_k.

The unique feasible allocation satisfying those inequalities is ϕ. ■

The next example illustrates that, in general, the profile of strategies that forms a MBRE is not subgame perfect equilibrium.

Example 5.4. *Let* $N = \{1,2,3,4,5\}$ *a set of players and consider the following 5-person veto balanced game* (N, v) *where*

$$v(S) = \begin{cases} 31 & \text{if } S \in \{\{1,2,3,5\},\{1,2,3,4\}\} \\ 26 & \text{if } S = \{1,2,4,5\} \\ 51 & \text{if } S = N \\ 0 & \text{otherwise.} \end{cases}$$

Computing the outcome associated to any MBRE we see that the proposer receives as payoff the amount $\phi_1(N, v) = 60.5/3$. As we know this result is true for any order. Suppose now the following order in the set of responders: The first responder is player 2, the second player 3 and the last one is player 5. The following result holds given this order.

If the responders play optimally the game (not necessarily as myopic maximizers) the proposer can get a payoff higher than the one provided by the MBRE outcome. Therefore MBRE outcome and SPE outcomes do not necessarily coincide.

The strategy is the following: The proposer offers nothing in the first three stages. In the 4th stage the proposal is: $(10, 10, 5, 0, 0)$.

The answer of player 2, 4 and 5 does not change the proposal (even if the proposal faced by player 4 and 5 is a new one resulting from a rejection of the player 3). If player 3 accepts the TU game for the last stage will be:

$$w(S) = \begin{cases} 11 & \text{if } S \in \{\{1,2,3,5\},\{1,2,3,4\}\} \\ 11 & \text{if } S = \{1,2,4,5\} \\ 26 & \text{if } S = N \\ 0 & \text{otherwise.} \end{cases}$$

In the last stage everybody play as myopic maximizer and the outcome should be an element of $B_1(N, w)$. It can be checked that $B_1(N, w) = \{(5.5, 5.5, 0, 0, 0)\}$. Therefore after accepting the proposal player 3 gets a total payoff of 5.

In case of rejection, the TU game for the last stage will be:

$$u(S) = \begin{cases} 11 & \text{if } S \in \{\{1,2,3,5\},\{1,2,3,4\}\} \\ 6 & \text{if } S = \{1,2,4,5\} \\ 26 & \text{if } S = N \\ 0 & \text{otherwise.} \end{cases}$$

As before, in the last stage everybody play as myopic maximizer and the outcome should be an element of $B_1(N,u)$. It can be checked that $B_1(N,u) = \{(5.2, 5.2, 5.2, 5.2, 5.2)\}$. Therefore after accepting the proposal player 3 gets a total payoff of 5.2.

Therefore a rational behavior of player 3 implies a rejection of the proposal in the 4th stage. This rejection is not a myopic maximizer´s behavior. After rejection of player 3 the proposer gets a payoff of 20.2, higher than 60.5/3.

In this game the outcome associated to MBRE is not the result of a SPE.

References

[1] Arin J and Feltkamp V (1997) The nucleolus and kernel of veto-rich transferable utility games. Int J of Game Theory 26:61-73

[2] Arin J and Feltkamp V (2001) Monotonicity properties of the nucleolus in the class of veto balanced TU games. Mimeo.

[3] Dagan N., Volij O. and R. Serrano (1997) A noncooperative view on consistent bankruptcy rules. Games and Economic Behavior 18, 55-72

[4] Maschler M (1992) The bargaining set, kernel and nucleolus. Handbook of game theory with economic applications I, Aumann R. J. and Hart S. eds., Amsterdam: North-Holland

[5] Meggido N (1974) On the monotonicity of the bargaining set, the kernel and the nucleolus of a game. SIAM J of Applied Mathematics 27:355-358

[6] Schmeidler D (1969) The nucleolus of a characteristic function game. SIAM J of Applied Mathematics 17:1163-117